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Estimation of tail risk and portfolio optimisation with respect to extreme measures

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Abstract

In this work we discuss in detail the theoretical and practical issues related to the adoption of a Poisson-Gaussian probability measure of financial returns as reference model for risk estimation and control during periods of market instability. The problems posed, as a consequence of its lack of coherence, by the adoption of the *Value-at-Risk* (VaR) for risk measurement and control purposes are considered in general and with respect to a case study represented by the Argentinean 2001 crisis. The generalisation of the canonical Gaussian framework to account for possible market instabilities and the role that can be played by portfolio optimisation schemes as form of control of extreme downside risk, are related in the article to the current debate on the necessity to overcome the problems posed by VaR in risk management practice. The introduction of coherent risk measures in portfolio optimisation and their effectiveness in risk management is considered by allowing arbitrary assumptions on the probability measure governing financial returns. A heuristic method to introduce a shock propagation effect in a multivariate Poisson-Gaussian framework is also proposed and tested in the given case study. A rich set of numerical and computational evidences supports the arguments raised in the study.

JEL classification: G11; C13; C44
Keywords : Poisson-Gaussian probability measure ; Extreme values ; Value-at-Risk; Expected shortfall; Convex risk measures; Risk control; Portfolio optimisation; Shocks propagation

1. Introduction

Following the BIS 1995 ruling[2], *Value-at-Risk* (VaR) has become the industry standard to define the market risk exposure relevant for capital adequacy calculations. Defined as the maximum potential loss at a given confidence level, typically 95% or 99% and for a given horizon, this measure has attracted increasing interest in the scientific community after the seminal article by Artzner et al (1997, 1999), which opened the way to the axiomatic risk theory and enhanced the effort to overcome both practical and theoretical drawbacks of this measure (see Pflug, 2000, Acerbi, 2002, Szegö, 2002, Tasche, 2002 to mention just few recent contributors).
The adoption of VaR and other tail measures in financial practice has paralleled in the last few years an increasing effort to include these measures within portfolio optimisation schemes. An effort that has significantly enriched the debate and more generally extended the scope of risk management (RM). Portfolio optimisation provides in this context a natural framework to assess the potential of introducing specific risk measures in a risk control perspective. The introduction of extreme risk parameters in portfolio optimisation models and their estimation has proven critical particularly during periods in which financial markets, as frequently observed in the recent past, have been subject to sudden down movements resulting in corresponding deviations from canonical normal and lognormal market assumptions. Just in the last two years: the 2002 US equity market crisis followed the Argentinean crisis of July and October 2001 which followed the Dot.com bubble explosion of April 2000, and so on.

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[2] Bank of International Settlements, December 1995: Communiqué announcing an amendment to the Basle Committee on Banking Supervision. BIS Review no. 209.

These events have not only severely undermined the financial solidity of domestic and global investors and many investment houses, but have produced evidence of the relevance of extreme events suck as market crises in RM practice.
Two issues become central:

- How accurate is the method employed to estimate the profit and loss (p&l) distribution under those market conditions, and
- Whether any strategy can be undertaken by the decision-maker (the risk manager) in order to control the portfolio downside when these events occur.

In what follows we intend to discuss in detail the underlying methodological implications of abandoning generally accepted assumptions of mainstream finance – such as returns normality – and link the debate on risk measures estimation to the issue of effective risk control.

Risk measurement has to do with the adoption of methods able *ex-ante* to measure accurately the risk of a financial position. The choice of the measure is clearly part of the issue.
In practice effective risk measurement amounts to the identification of the appropriate parametric or non parametric approximation of the portfolio p&l distribution: this in turn leads to the estimation of a given set of risk parameters, such as the VaR, at given confidence intervals.
Alternative assumptions on the probability measure will result in different risk measurement methods.
The estimation of a parametric or non parametric distribution, requires as first step the portfolio full or approximated revaluation on a data history of associated risk factors: the resulting constant-weighted portfolio returns sample is the fundamental input of the risk measurement step.

Risk control, instead, has to do with the actions to be undertaken in order to modify the portfolio risk profile and induce *ex-post* a more desirable behaviour of the p&l.
Different techniques can be adopted to effectively control the risk of a possibly complex portfolio structure. Hedging strategies based on option contracts have historically played this role in modern finance: in many circumstances however these are not practical either because total lack of liquidity in the associated markets or because far too expensive. Partly for this reason and to a larger extent because of the more comprehensive approach they allow, portfolio optimisation methods based on the explicit introduction of portfolio tail risk measures in portfolio selection schemes are gaining grounds as viable methods to maintain a grasp on the portfolio downside.
Under the normality assumption on portfolio returns, RM calls for the adoption of the variance covariance method. This quite clearly matches the mean variance approach to portfolio selection (Markowitz, 1952). JP Morgan indeed proposed the variance covariance approach to Value-at-Risk estimation, relying on the same assumptions that motivated in the first place Markowitz theory.
Several approaches have been proposed in the literature, pretty much independently, to achieve an accurate estimation of risk measures in the tails, on one hand, and include such measures within newly established portfolio optimisation paradigms.
The debate on risk accurate estimation goes hand-in-hand with the issue of the risk measure coherence. Following Artzner et al (1999) and subsequent relevant contributions by Acerbi and Tasche (2001), the lack of the sub-additivity property of VaR leads to a series of highly undesirable features of this measure. The expected shortfall, defined as the conditional expected value beyond a certain threshold, which is sub-additive and belongs to the general class of spectral measures (Acerbi, 2002), can overcome most of the problems posed by the VaR measure.
In what follows we consider the need to go beyond the VaR, still maintaining, as required by the current industry standards, this measure within the bulk of our discussion.
The article has the following structure.
In section 2, we summarise the methodological steps linking the assumptions on the probability space adopted for risk assessment, to the associated portfolio control problem. In section 3 we focus on RM methods particularly suitable during periods of financial instability. The relevance of the risk measure convexity properties becomes in this setting even more critical.
In section 4 we revise and apply to a specific case study the portfolio optimisation approaches that, under arbitrary assumptions on the probability space, can be applied to control extreme risk and lead to effective portfolio strategies under rapidly changing market conditions.

2. From risk measurement to risk control: the set-up

A risk measure ρ can be simply defined as a certain mapping from a given probability space, where the portfolio value function $V(x,\omega)$ is defined, into the reals $\rho : X \times \Omega \to \Re$. $\omega \in (\Omega, \Xi, P)$ is a random variable and $x \in X$ is a given portfolio position. The triple (Ω, Ξ, P) specifies the sample space, the sigma-field and the probability measure characterising the probability structure.

The definition of the VaR at the horizon T for a given confidence level α is defined by the maximum tolerable loss associated with the current portfolio position:

$$VaR_{\alpha}^{T} = \inf\{l \in \Re : P(-\Delta V_T \geq l) \leq 1-\alpha\} = \inf\{l \in \Re : F_{-\Delta V_T}(l) \geq \alpha\} \qquad (1).$$

In (1) the variable l defines the loss, while the change in portfolio value between 0 and T is denoted by $\Delta V_T = V_T - V_0$.

Definition (1) considers an aggregate VaR at the horizon T. The risk factors driving the profit and loss (p&l) behaviour are typically identified by decomposing the aggregate value into its constituent parts. For $j = 1,...J$, with $\Delta V_t = V_{t+1} - V_t$, we have:

$$\Delta V_t = \sum_{j=1,2,..,J} \Delta V_t^j = \sum_{j=1,2,..,J} V_t^j (1+\omega_{t+1}^j) \qquad (2),$$

where ω_t^j, for $t = 1,2,..$, is a random return in market j, assumed to be the relevant risk factor for risk assessment.

Different assumptions on the probability measure in (1) will result in different methods for risk assessment:

- the empirical, frequency distribution is associated with the historical simulation method,
- the joint normal distribution is associated with the variance covariance method originally proposed by JP Morgan in RiskMetrics (1995),
- alternative parametric approximations based for instance on extreme value distributions, normal mixtures or elliptic distributions, can also be employed for VaR estimation, as in McNeil (2001), Duffie (1999), Eberlein (2001).
- More generally, arbitrary assumptions on the probability measure will require the use of Monte Carlo methods

In the following section we consider two methods, expressively constructed in order to take appropriately into account the possibility of events incompatible with canonical assumptions on market behaviour.

Extreme value theory (EVT) has played an important role in the extension of risk measurement techniques to account for rare events such as financial crises, requiring the adoption of specific statistical methods and the introduction of results coming from asymptotic probability theory. Developments in this framework are currently considered by a number of financial institutions. We revise the main steps of an EVT based risk management approach along with an application to a recent financial crisis in § 3.

An alternative approach, also discussed in the third section, that tries to overcome the estimation problems posed by limited sample sizes, relies on the adoption of a mixed Poisson-Gaussian probability measure, with market shocks and instability periods captured by the Poisson process. The properties of this approach are also considered with respect to a case study.

The opportunity to base the choice of a risk assessment methodology relying on statistics on the dynamic behaviour of the technique are considered in § 3.

Out-of-sample back-testing is typically employed in order to study the properties of the violations observed with respect to previously estimated risk parameters such as the VaR (see Consigli, 2002 and section 3 here below). *A posteriori*, indeed, once the return is realised, the change in value $\Delta V_t^j = V_{t+1}^j - V_t^j$ is known and the accuracy of the risk estimation can be assessed. Back-testing allows a ranking of the different methods over the given period. It has been argued that the validity of one RM method can to a certain extent depend on the current market regime (unstable, downsided, fat tailed, etc.). In general a method that *on average* provides the best approximations tend to be preferred.

Portfolio positions will typically change in (2) between t and $t+1$. Let then x_t^j and p_t^j denote the number of contracts and corresponding pricing function at time t for unit *j*, then $V_t^j = \sum_j x_t^j p_t^j$ and we see that ex-post $\Delta V_t^j = \sum_j (x_{t+1}^j p_{t+1}^j - x_t^j p_t^j)$.

The change in position $x_t^j \to x_{t+1}^j$ is unaccounted for when the risk measure is estimated in the first place and it does define the control variable in the hands of the risk manager to limit the potential downside.

We are interested in that decision process $x \in X \subseteq \Re^n, x := \{x_t\}_{t=1}^T$ that minimises a risk measure such as (1) under the same assumptions introduced in the RM application. The general form of the optimisation problem is

$$\min_{x \in X} \rho(x, \omega) \quad s.t. \;\; g(x, \omega) \in Q \qquad (3),$$

where $\omega \in (\Omega, \Xi, P)$ is a vector random variable, ρ is a risk functional, $g : X \times \Omega \to Q$ is a constraint matrix operator mapping into a convex set Q.

In the familiar RiskMetrics setting, P is the joint normal distribution, the VaR at the 99% percent for the portfolio x, with one day horizon, is $VaR_{99\%} = -2.3624\sigma_x - \mu_x$, with μ_x for the expected return of the portfolio and $\sigma_x = [x'\Sigma x]^{1/2}$ for its volatility. In the RiskMetrics methodology, the variance covariance matrix is estimated adopting an exponentially weighted moving average (Ewma) method. The variance of portfolio returns is also the definition of risk proposed originally by Markowitz (1952) yielding under the same assumptions, the celebrated formulation of the portfolio selection problem:

$$\min_{x \ge 0} x'\Sigma x \qquad s.t. \;\; r'x = \mu, \quad 1'x = 1 \qquad (4).$$

Problem (4) has a quadratic objective and a set of linear constraints and can be solved as a quadratic programming problem (QP) with several available algorithms (see Luenberger, 1989).

In the following two sections we consider other possible approaches to risk measurement and control when different assumptions on the probability measure for risk assessment are considered and risk measures other than the variance are introduced in (4). The departure from the Gaussian distribution is primarily motivated by the search of a more accurate fitting of market returns, leading to improved risk estimates. It turns out (see § 4) that the introduction of risk measures alternative to the variance within a non Gaussian framework, can easily be handled within a linear programming (LP) formulation and, however, if the VaR, instead of the variance is considered, not only we introduce in the problem a non coherent measure but we also make the portfolio selection problem hardly solvable, as explained next. Indeed, the solution of (3) relies in general on the convexity with respect to x of both the objective function and the constraints: this is, together with its lack of sub-additivity, one of the problems posed by the adoption of VaR as reference risk measure (see Gaivoronski and Pflug, 2000, Rockafellar and Uryasev, 2000, Acerbi, 2002, Consigli, 2002).

2.1 VaR control with non-normal return distributions

Lack of convexity (Rockafellar, 1970) of the VaR measure with respect to portfolio positions under non-normal return distributions implies that given two portfolios $x_1, x_2 \in X$, it might occur that taking ρ in (3) as the VaR mapping we have $\rho(\lambda x_1 + (1-\lambda)x_2) > \lambda\rho(x_1) + (1-\lambda)\rho(x_2)$ (5).

In the risk theory *language* (Artner et al,1997, 1999), (5) implies that the VaR violates the sub-additivity -- $\rho(x_1 + x_2) \le \rho(x_1) + \rho(x_2)$ -- and positive homogeneity -- $\rho(\lambda x) = \lambda\rho(x)$ -- properties. Or in more traditional mathematical terminology, Jensen inequality does not hold.

From (5), not only an incentive to risk concentration can follow from the adoption of the VaR measure, but also if we try to control the VaR risk by combining our portfolio positions we may end up worse off.

This evidence is documented in several contributions and recently in Consigli and Di Cesare (2001) and Grootveld and Hallerbach (2001 and this volume). In Consigli (2002) with reference to a fund manager relative optimisation problem again under Poisson-Gaussian uncertainty, and in Basak, Shapiro and Teplà (2001) in a more general setting.

We give a graphical representation of the problem in figure 1, where the variance profile is plotted against the VaR(99%) risk measure as function of two assets portfolio.

In the figure we show on the left plot the variance and on the right plot the VaR at 99% estimates as functions of a portfolio structure changing from a fully Argentinean bond to a fully US equity invested portfolio.
Both risk measures where estimated on a portfolio space generated across the July 2001 Argentinean crisis by Monte Carlo simulation, assuming an underlying jump-diffusion return process (see § 3). We will considered this case study again in § 3 and § 4. Now, we are only interested to point out the behaviour of the VaR function.

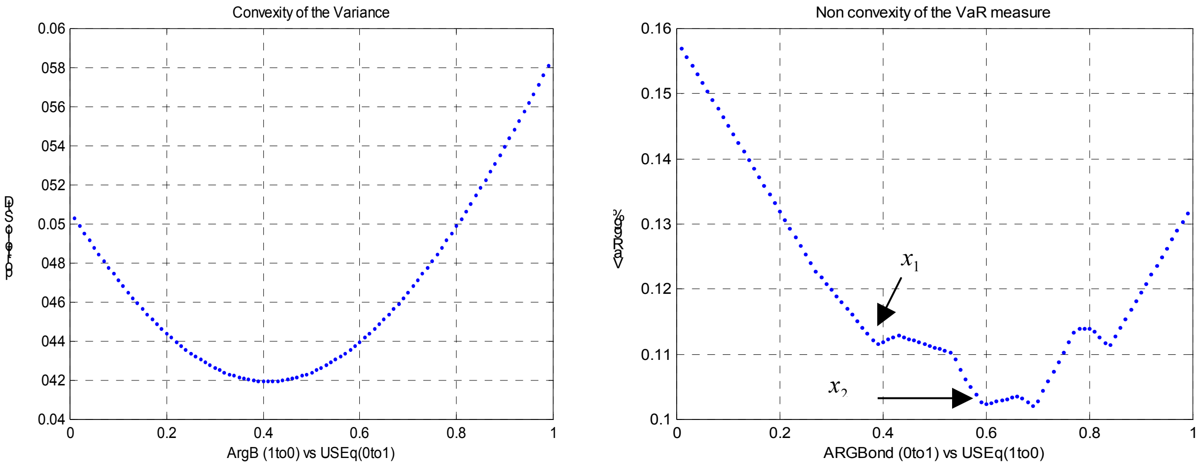

Figure 1. Replacing variance (left) with VaR99% (right) in portfolio selection

The inclusion of the VaR as risk measure leads to the problem non-convexity in the general case of non-Gaussian return distributions.
From (1) we also see that the problem of the VaR non convexity is critical in the low variance, low VaR region, as also recalled by Frey and McNeil (2002).
On the right plot of figure 1, a non convex, non smooth region can be found for instance between portfolios x_1 and x_2, with $\rho(\lambda x_1 + (1-\lambda)x_2) > \lambda\rho(x_1) + (1-\lambda)\rho(x_2)$.
In figure 2 we present the entire risk-return profile of a portfolio space in the mean-volatility-VaR(99%) space defined by swapping from the full emerging bond portfolio to the full equity portfolio.

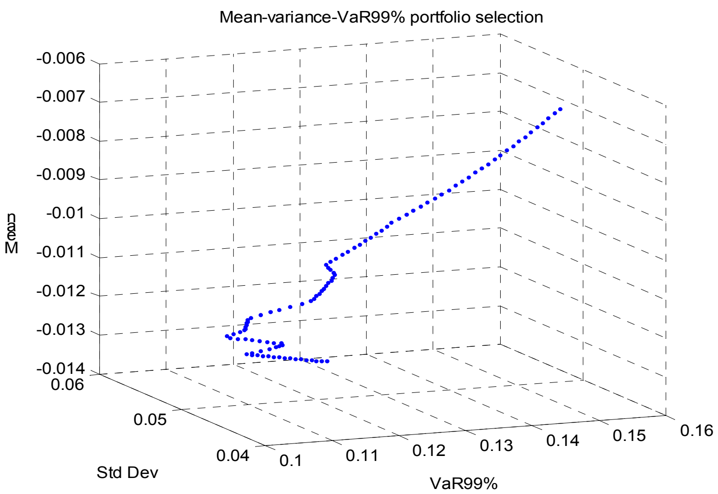

Figure 2. Example of Mean-variance-VaR(99%) optimisation across the 2001 Argentinean crisis

A three-dimensional approach to portfolio selection was proposed by Consigli and Di Cesare (2001) and independently by Wang (2000). Both suggest the joint consideration of variance and VaR as reference risk measures. The mean-variance and mean-VaR problems represent in this case sub problems of the general multidimensional case.

Under non normality, we can have mean-variance efficient portfolios carrying different VaR estimates and portfolios with a given VaR estimate and different variance attributes.
This evidence is considered in Wang (2000), who proposes a 2-steps optimisation procedure to solve the problem: several problem-types do result as a consequence.
He suggests a solution approach based first on the solution of a 2-dimensional risk-return problem with respect to one risk measure, followed by an unconstrained minimisation of the second risk attribute.
Consigli (2002) proposes a heuristic simulation approach to mean-variance-VaR optimisation defined on the whole portfolio space, relying on the direct estimation of the return and risk coefficients in a Monte Carlo framework.
The set of efficient portfolios with respect to the extreme risk measure is found in the proposed method, with a given tolerance on the variance, by recursively searching for the minimum risk as the expected return increases. A globally convergent algorithm needs to be applied at this point.
Any probability distribution and any risk measure constructed thereupon can be accommodated in this framework. This procedure yields an optimal decision. As shown in figure 2, however, in presence of the VaR measure a discontinuous efficient frontier will in general be generated. This suggests the introduction of a coherent risk measure, defined in the tail as well, replacing the VaR, or a convenient convex approximation of the VaR.
This can be done through the inclusion of a minimal *majorant* (Tasche, 2002) measure of the VaR (if this is to be maintained as standard tail measure) such as the expected shortfall with respect to the VaR (see also the results reported in table 3 below).
In order to overcome the problem posed by the VaR non convex and non smooth behaviour, several proposals have been put forward in the last years.
Gaivoronski and Pflug (2000) address the problem directly proposing an outer convex approximation of the VaR measure, that would lead to a convex problem: the application of usual convex algorithms will then result in a global optimum.
Rockafellar and Uryasev (2000) propose the replacement of VaR with the conditional VaR, or CVaR, the expected value beyond the VaR, a coherent and convex measure in presence of continuous distributions, which also induces an indirect constraint on the VaR. The method, which is based on the introduction of an auxiliary variable in the optimisation procedure, allows the adoption of a linear programming solution method. As shown in § 4, this is extendible in general to risk-return methods assuming the measure convexity with respect to portfolio rebalancing. We recall other LP solvable risk-return problems in § 4.1. By removing the VaR from the risk-return trade-off, not only we do avoid the problems posed by its lack of coherence but we also gain in computational tractability.
Acerbi and Simonetti (2002) propose an optimisation procedure based on a risk measure taken from the general (coherent) class of spectral risk measures. The expected shortfall is a noticeable example of such a case. As discussed below, by clarifying that a convex combination of coherent risk measures (providing in case an 'optimal' VaR approximation) leads to again to a convex measure, they suggest the possible adoption of subjectively constructed risk measures.
All these approaches propose a risk-return optimisation procedure in two dimensions, that induce a control on the VaR by introducing an alternative desirable measure. We will elaborate further on this in section 4.

3. Beyond VaR: from non coherent to coherent measures

It is now more than six years since the publication of the first version in 1997 of the article on risk measures' *coherence* by Artzner et al (1997), that the need to overcome the VaR as standard for RM purposes is under discussion. Several measures have been suggested as remedies (Szegö, 2002), the expected shortfall resulting at present as the preferable. Following Acerbi (2002), this can be regarded as a special specification of a more general class of coherent risk measures, the family of spectral measures. We'll get back to this later on.
The expected shortfall is defined as the expected value beyond a given shortfall, it is the mean of a shortfall probability distribution. By conditioning on the VaR at a given confidence level it does coincide, in presence of continuous distributions, with the so called Conditional VaR (CVaR) introduced by Rockafellar and Uryasev in (2000) (again the minus in front of the portfolio value change is included to make the losses positive):

$$ES_{\alpha}^{T} = VaR_{\alpha}^{T} + E\left[-\Delta V_T - VaR_{\alpha}^{T} \mid \Delta V_T \leq -VaR_{\alpha}^{T}\right] = E\left[L_T \mid L_T \geq VaR_{\alpha}^{T}\right] \quad (6),$$

where we defined the loss at time T as $L_T := -\Delta V_T$.
In its more general form, valid for continuous as well as discrete distributions according to Acerbi and Tasche (2001) , the expected shortfall is given by:

$$GES_\alpha^T = \frac{1}{1-\alpha}\left[E(L_T; L_T \geq VaR_\alpha) + q_\alpha(1-\alpha - P(L_T \geq VaR_\alpha))\right] \qquad (7).$$

As in Frey and McNeil (2002), we call this measure the generalised expected shortfall (GES). The second quantity on the RHS is 0 in absence of discontinuities, $1-\alpha = P(L_T \geq VaR_\alpha)$, it is non null otherwise.
In spite of its properties (Acerbi, 2002) the expected shortfall is failing to replace the VaR as industry standard for RM. Several reasons can be mentioned for this lack of progress. The debate is considered in detail in Consigli (2003).
Here below we touch this issue recalling the problems posed in practice by the VaR measure and presenting numerical results in which both quantities are considered.
The CVaR or the ES are risk measures expressively constructed in order to monitor the behaviour of tail returns. Other measures have been suggested in the literature to account for downside risk and derive within portfolio selection models optimal policies with respect to specific risk parameters. Deviation measures as those considered in § 4, are such an example.
The tail conditional expectation, TCE_α defined as a tail expected value with respect to a given confidence, and the worst conditional expectation, WCE_α the expected value over the worst possible losses, have also been considered as relevant measures for extreme risk control (see Acerbi and Tasche, 2001). It turns out however that the first measure is in general not coherent, while the second may result unstable in presence of discontinuous distributions, for small changes of α, a problem which is not posed by the expected shortfall. Both the TCE_α and the WCE_α can be accommodated within LP-solvable portfolio representations.
For continuous distributions both the WCE_α and the ES_α on the right confidence level provide minimal majorants of the VaR. They can thus provide the required approximating coherent risk measure.
We benchmark in the final part of the article the optimal strategies induced by different portfolio paradigms applied to the Argentinean case problem.

3.1 Risk measurement in the tails: methods accuracy

In Consigli (2002) was argued that the VaR shortcomings become even more relevant under conditions of markets instability. Furthermore the importance to address the issue of accurate risk measurement together with, and even before than, the issue of a suitable risk measure was also pointed out. Here we continue in that direction, providing more insight on the related analytic and theoretical implications.
Starting point of the analysis is the evidence that, contrary to conventional financial theory, empirical evidence suggests that market-oriented financial systems are prone to periodic crises.
Nine major financial crises with various impact have been recorded during the 90's: the western European exchange rate mechanism crisis in 1992, the Mexican crisis in 1994-95, again the East Asian crisis in 1997-98, the Russian crisis and the LTCM Hedge fund crisis in 1998, the Brazilian crisis early in 1999, the April 2000 Dot.com US equity crisis, the 2001 Argentinean crisis and last the sequence of shocks registered in the US market after the 2001 September terrorist attack and during the year 2002.
Each such event can be seen from two different perspectives: as single events modelled as realisations of a Poisson process, or as rare events generated by an extreme value distribution (EVD). In both cases we are in search of an appropriate model for financial events that, under any other model would be given, if any, a very negligible probability of occurrence.
The first approach was considered in Consigli and Di Cesare (2001), where the financial crisis induced by the risk of a widespread Russian default on foreign debt in 1998 was modelled by the arrival of an inhomogeneous Poisson process.

Similarly, an accurate probability assessment of these rare events and, as a consequence, an accurate estimation of the portfolio risk, can be achieved by fitting on the tail only a limit distribution such as a GPD (see Embrechts 2000, Consigli et al. 2001).
In the first case we do introduce a mixed probability distribution on the full support of the return function. In the second the parametric conditional distribution is generated from a sample set containing only tail events. Both approaches appear suitable to estimate extreme percentiles of the portfolio return distribution in presence of market instability. As further discussed in the following sections, however, the former allows an extension of the probability assumptions underlying tail estimation into portfolio optimisation and in general direct risk control.
In both, EVT and mixed Poisson-Gaussian cases, we are able to approximate the tail frequency distribution with an accurate continuous parametric approximation.
Two statistical evidences are widely quoted to explain from a statistical viewpoint the inadequacy of the Gaussian distribution as model of financial returns:

- Daily returns, not just for equity markets but increasingly for high yield bond and currency markets, are subject to frequent changes of volatility regimes,
- In many cases return distributions do appear not only leptokurtic (i.e. fat-tailed) but also skewed, and sometime even quite humpy! This is typically the case when a severe shock or sequence of shocks has been observed in the market. Credit portfolios (Kijima and Muromachi, 2000) are particularly exposed to events clustering in presence of corporate defaults with homogeneous recovery rates.

Financial analysts and scientists, as a result, have started to abandon the Gaussian model, moving towards the use of alternative approaches and EVT techniques in particular for the estimation of extreme market risks.

3.1.1 EVT-based risk management

In the following discussion we consider the implications of a pure EVT approach and analyse its features independently of the introduced assumptions on the core of the return distribution.
From its original formulation (see as reference volume Embrechts et al, 1997) EVT applications in insurance and finance have gained enormous support as they were conceived to cope expressively with events occurring far in the tails of the reference distributions. From a practical point of view, the (parametric) continuous limit distributions estimated within this framework were expected to provide a stable fitting (with respect to realised extreme events) and thus a valid reference distribution for capital adequacy assessment and *limit VaR* definition.
Once an appropriate tail fitting is achieved and the parameters of the extreme value distribution (EVD) estimated, we can stress test the distribution stability.
Two results provide the theoretical grounds for the adoption of an EVT approach.
The first result, known as the Fisher-Tippett theorem (1928) clarifies under which conditions we can expect that a limit distribution belongs to the class of extreme value distributions. The second, credited to Balkema and de Haan (1974) and Pickands (1975), motivates the adoption of the *Generalised Pareto distribution* (GPD) as reference distribution in EVT applications.
The Fisher-Tippett theorem can be regarded as the relevant central limit theorem for the class of EVD. The use, in this set-up, of the GPD function is then justified by the fact that if, under the same assumptions, the distribution of the *excesses* $F_u(x) = P[X - u \le x \mid X > u], x > 0$ beyond a certain threshold u is such that an EVD exists, than it can be proven that this limit distribution is the two parameter GPD $G_{\xi,\beta(u)}(x)$.
From a practical point of view the use of this approach requires the identification of a sufficiently high threshold to justify the application of Pickands theorem and the use of a set of statistical tests to make sure that we are in the domain of attraction of a limit distribution. Among these statistical tools (see also the case study below) the behaviour of the sample mean excess function plays a specific role (see Embrechts et al 1997). This is defined as the mean over the excesses with respect to a certain threshold:

$$e_n(u) = 1/n \sum_{i=1}^{n} (X_i - u)^+ \qquad (8)$$

where n is the assumed sample of returns $(X_i - u)^+$ above the threshold u.
The sample mean excess function in (8) is an estimate of the mean excess function $e(u) = E[X - u \mid X > u]$, where X are daily returns. The interpretation of the mean excess function is explained in (Embrechts et al., 1997). If the points of the sample mean excess show an upward

trend then this is a sign of heavy tailed behaviour. If furthermore, above a threshold, the mean excess follows a straight line, then this suggests that the excesses above that threshold are distributed according to a GPD.

From the selected threshold a limited sample of extreme returns $\zeta \in GPD(\xi,\beta), \zeta = \{\omega : \omega \le u\}$ is identified, allowing the estimation of the GPD parameters. The GPD density:

$$f(\zeta) = \beta^{-1}(1+\xi\frac{\zeta}{\beta})^{-(\xi^{-1}+1)} \qquad (9)$$

induces in particular the following log-likelihood maximisation problem

$$\max_{\xi,\beta}\left\{L((\xi,\beta);Z) = -N^{exc}\ln\beta - (\xi^{-1}+1)\sum_{i=1,..N^{exc}}\ln(1+\frac{\xi}{\beta}Z_i)\right\} \qquad (10).$$

In (10) N^{exc} defines the number of excesses, or *exceedances*, and $Z_i = X_i - u$ are the individual excesses above the estimated threshold.

The parameters ξ and β are respectively the *shape* and the *scaling* factors of the distribution.

Unbiased estimates of the two parameters require a sufficient sample set for the estimation procedure: this is the crucial trade-off problem of any EVT applications. The threshold needs to be sufficiently far in the tail to apply the above mentioned EVT results, while the sample size needs to sufficiently large to allow a correct formulation and solution of the estimation problem[3].

A sample of around 60 excesses can in general be regarded as a lower bound for the definition of acceptable parameter estimates within a ML estimation procedure.

As further analysed in section 3.2 the solution of such a trade off problem characterises the application of EVT techniques to VaR estimation.

An EVT based algorithm for VaR estimation requires thus a number of steps, that we can briefly summarise:-

1. A series of daily returns on the current portfolio position needs to be collected. This is the canonical input to any technique for probability estimation,
2. The data are then sorted and the mean excess function evaluated iteratively as the threshold increases and the corresponding number of excesses decreases,
3. Once the threshold for the fitting of the GPD is identified and the corresponding sample of excesses delimited,
4. The parameters of the GPD can be estimated, allowing the definition of the continuous tail distribution.
5. Upon which the required set of risk parameters, such as the VaR or the expected shortfall, can be derived.

The GPD is fitted above a certain threshold. Following the discussion in the previous session, we can assume that above this threshold, for given ξ,β, $F_u(\zeta) = P[X-u \le \zeta \mid X > u] = GPD_{\xi,\beta}(\zeta)$.

On the set of non positive returns we will have, for $x = u + \zeta$, (see Mc Neil, 2000) a tail estimator given by

$$F(x) = 1 - \frac{N^u}{n}(1+\xi\frac{x-u}{\beta})^{-\xi^{-1}} \qquad (11).$$

In (11), n is the complete sample set and N^u is the number of excesses above *u.*

From (11), for a given probability $\alpha > F(u)$, we can derive the 1 day VaR(α %) estimate as

$$VaR_\alpha = u + \frac{\beta}{\xi}\left[\left(\frac{n}{N^u}(1-\alpha)\right)^{-\xi} - 1\right] \qquad (12).$$

[3] It can be shown (see Hosking and Wallis 1987) that for ξ > -0.5 ML regularity conditions are fulfilled and that the maximum likelihood estimates of the two parameters of the GPD are asymptotically normally distributed: $\begin{pmatrix}\hat{\xi}_n \\ \hat{\beta}_n\end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix}\xi \\ \beta\end{pmatrix}, \begin{pmatrix}(1+\xi)^2 & \beta(1+\xi) \\ \beta(1+\xi) & 2\beta^2(1+\xi)\end{pmatrix}\right]$.

The extreme value GPD, thus the VaR in (12), is, by construction, more accurate than the other methods on the tails only, and furthermore (see the Argentinean case-problem below) is more stable with respect to the set of market extremes.
Given (12) we can also derive the expected value of the shortfall distribution, here delimited by the estimated VaR. Assuming a shape below 1, it can be shown (McNeil, 2000) that the ES at the given confidence interval is

$$ES_\alpha = \frac{1}{1-\xi} VaR_\alpha + \frac{\beta - \xi u}{1-\xi} \qquad (13).$$

The relevant parameters and risk measures are typically derived in the EVT approach by analysing a one dimensional series, corresponding for instance to the p&l of a business unit. Once the corresponding risk measures have been identified, the aggregate risk measure needs to be computed.

3.1.2 The Poisson-Gaussian model

We consider now a return process defined in a probability space with Wiener and Poisson uncertainty:

$$\frac{dV_t}{V_t} = \mu(t)dt + \sigma(t)dz_t + \rho(\omega)dN_t(\lambda) \qquad (14).$$

In (14) the instantaneous return $\frac{dV_t}{V_t}$ over *dt*, with $dV_t := V_{t+dt} - V_t$ and V_t the portfolio value at time *t*, can be approximated by $\frac{dV_t}{V_t} \approx \ln \frac{V_{t+dt}}{V_t} = v_{t+dt} - v_t = dv_t = \omega_{t+dt}$. In the simple constant coefficient case, model (14) reduces to the seminal model of Merton (1976).
The process is characterised by an instantaneous, possibly time-varying drift $\mu(t)dt$, a volatility term $\sigma(t)dz_t$ -- where $dz_t \sim N(0, dt)$ is a Brownian Motion process – and a homogeneous random (in time and space) Poisson increment $\rho(\omega)dN_t$.
The Poisson process $N_t \sim Poisson(\lambda)$, $dN_t := N_{t+dt} - N_t$ has a mark ρ -- the *magnitude* of the *jump* generated by the Poisson arrival – that in Merton (1976) was assumed to be lognormal: we also assume $\rho \approx \ln(\rho+1) \sim N(\vartheta, v^2)$. The parameter λ of the Poisson process defines the mean number of shocks per unit time.
Under these assumptions the density of the return process can be exactly specified and maximum likelihood estimation performed (see Consigli, 2002, Ball and Torous,1985, Kim et al, 1994).
At any point in time, ω has density $\varphi(\omega)$ given by

$$\varphi(\omega) = \left[\sum_{k=0}^{\infty} \frac{e^{-\lambda\Delta t}(\lambda\Delta t)^k}{k!} N(\gamma\Delta t + k\vartheta, \sigma^2\Delta t + k v^2) \right] \qquad (15),$$

with $\gamma := \mu - \sigma^2/2$ and k denoting the number of shocks modelled through the Poisson component.
From (15), given that $P(N_{t+\Delta t} - N_t = 1) = \lambda\Delta t$ and given the lognormal distribution of the jump magnitude, we can see that the instantaneous mean and variance of the process are $E(\omega) = (\gamma + \lambda\vartheta)\Delta t$ and $V(\omega) = \left[\sigma^2 + \lambda(\vartheta^2 + v^2)\right]\Delta t$ respectively. The distribution is more peaked than the distribution of a comparable normal, and depending on ϑ it can be skewed and asymmetric.
Unlike the Wiener noise, that affects the evolution of the return process continuously in time, a jump is generated in (13) conditionally on the occurrence of a Poisson event. We assume the independence between the Wiener and the Poisson increments and between the arrival process and the jump measure associated with the event.
A generalisation of model (14) to account for possible time dependence of the Poisson intensity can be found in Consigli and Di Cesare (2001), where the parameter of the Poisson process is defined as a function of an implied instantaneous forward spread rate.
The use of a jump variable to capture the impact on high-yield bond markets of changing default probabilities has been proposed by several authors (see Jarrow and Madan 1995 as early reference, Das 1999 more recently). The adoption of these models for VaR estimation is also recent (see Duffie and Pan, 2001).

Relying on (15) we can estimate by maximum likelihood the parameters of the distribution, by solving the log-likelihood maximisation problem

$$\max_{\{\gamma,\sigma^2,\lambda,\vartheta,\nu^2\}} \{L(x_t;\gamma,\sigma^2,\lambda,\vartheta,\nu^2)\}$$
$$L := \sum_{t=1,\ldots,T} \ln\left(\sum_{k=0,\ldots,K} \left[\frac{e^{-\lambda\Delta t}(\lambda\Delta t)^k}{k!} \frac{1}{\sqrt{2\pi(\sigma^2\Delta t + k\nu^2)}} \exp\left\{ -\frac{[x_t - (\gamma\Delta t + k\vartheta)]^2}{2(\sigma^2\Delta t + k\nu^2)} \right\} \right] \right). \tag{16}$$

Problem (16) is solved below with a data input of 300 daily returns and truncating the sum over *k* in (15) and (16) to *K*:=3 maximum daily jumps.
The significance of the Poisson component is tested with a likelihood ratio test[4].
The solution of (16) in the five parameters $(\gamma,\sigma^2,\lambda,\vartheta,\nu^2)$ allows the specification of the mixed probability measure upon which tail risk computations can be based.
The model reduces to the standard Gauss assumptions when no Poisson events are allowed.
Observe that unlike the GPD likelihood function which is convex in the two parameters, typically the function in (16) will not be convex in all 5 parameters and the problems solution may result in a local maximum. In order to avoid this a good initial vector estimate – *hot start* -- is essential.
This is achieved in practice in three steps.
We first compute the mean, variance, skewness and kurtosis of the input return series. The first two moments define the input parameters for the mean and variance of the parameter vector while the skewness determines the sign of the mean jump size. The number of jumps are set equal to 10 and the jump size is initially set to 1%.
We then maximise the constant jump log-likelihood function with a 300 data series: this is now a 4 parameters model. The resulting optimal parameters provide the hot start to the jump diffusion model with a random jump magnitude. We use a quasi-Newton algorithm (see Luenberger, 1989) for non linear problems to solve (16).
The ML estimation of the process in (14) provides an input to a Poisson-Gaussian vector process in which each row represents a risk factor, as described in § 3.3.
First we test the validity of the Poisson-Gaussian approach against the EVT approach in a one dimensional set-up.

3.2 A case study. Application 1: risk measurement.

We can at this point apply the two methods described in § 3.1.1 and 3.1.2 to a case study coming from recent financial history: the Argentinean Eurobond market crisis recorded during the second week of July 2001. The representative index for this market is the JP Morgan total return index for Argentina in the class of the Emerging Market Bond Indices (EMBI).
Here we intend to benchmark different RM methodologies in a static analysis concentrated on a single event. A method validation can then be based on a dynamic back-testing study. The focus is on the ability of the two methods to overcome the limits shown by canonical RM techniques during periods of market turmoil.
During the second week of July, the market of Argentinean Eurobonds suffered a sequence of negative shocks, with a heavy impact on the market of high-yield bonds in general. On July the 12th the benchmark JPM Argentina felt 14% after a 5% drop the day before.
The following negative returns, furthermore, below –5%, have been recorded since June the 1st: July 11: -5.67%; July 12: -14.85%; July 27: -6.04%; July 31: -6.57%; August 16: -5.86%. This series of

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4 The idea behind the likelihood ratio test is to accept the null hypothesis: the Poisson parameter is insignificant – if the difference between two maxima of the log-likelihood function is small enough. The test statistic is:

$$\zeta_T^{LR} = 2\left[\max_{\mu,\sigma,\lambda} L_T(\Delta X;\mu,\sigma,\lambda) - \max_{\mu,\sigma} L_T(\Delta X;\mu,\sigma,0)\right].$$

It can be shown that under the null hypothesis $H_0 : (\lambda = 0), \zeta_T^{LR} \xrightarrow{d} \chi^2(1)$, with one degree of freedom.
This gives the critical value against which the value of the test statistic is to be compared.
In particular, for a 95% confidence interval, we have:

$$\text{Decision rule: } \begin{cases} \text{accept the null hypothesis} & \text{if } |\zeta_T^{LR}| \le \chi^2_{95\%}(1) \\ \text{reject } \textit{otherwise} & . \end{cases}$$

events did follow a period of limited volatility: the accuracy of the different methods under this scenario is seriously challenged.
Here below in Figures 3 and 4, we show the behaviour of the Argentinean market benchmark before and during the July crisis together with the empirical distribution generated at the end of July by 300 daily log-returns:

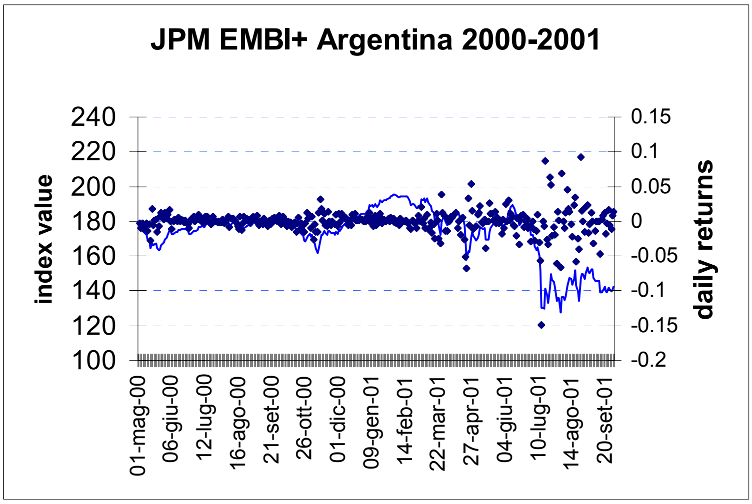

Figure 3. Argentina JPM EMBI+ behaviour May 2000—Sept 2001

The period of instability ended at the end of the year with the declaration of default on interest payments on US and Euro denominated debt. In November we observed another sequence of severe negative movements. At that point the market was already illiquid and political instability was spreading. Comparative optimisation results are presented in § 4 focusing on this second period.
From figure 3, it can be seen that, as already happened during the 1998 Russian crisis, the start of the crisis was indeed anticipated by a sequence of smaller market shocks. The market was to a certain extent anticipating the forthcoming instability. In figure 4 we report the frequency distribution generated by the set of 300 daily returns and the statistical estimates of the first four moments of the distribution, together with the minimum and maximum recorded 1-day returns.

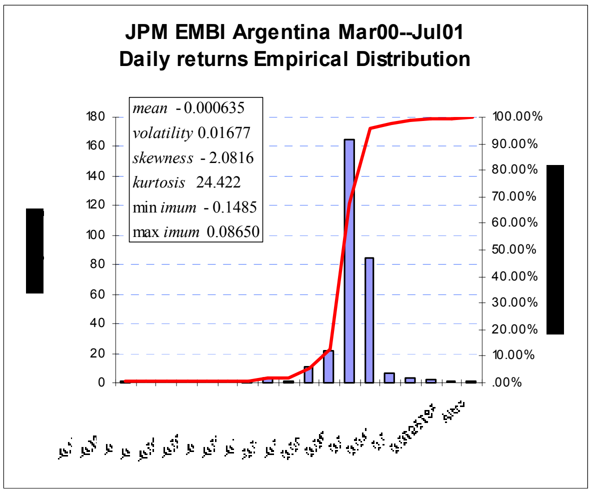

Figure 4. Argentina JPM EMBI+ empirical distribution as at the 30th of July 2001

The strong kurtosis and negative skewness reported in the figure are consistent with a market showing limited daily volatility, with heavy fat tails and asymmetry. The presented distribution can be considered as an input to an historical simulation (HS)-based RM application.
The EVT approach is based on the following three macro steps: threshold definition (thus sample set identification), GPD estimation and risk measures computation.

The first relies on the computation of the mean excess function and other preliminary data analysis. The graphical analysis of the QQ (quantile-quantile) and Hill plots deserve a mention.
In the first one, the quantiles of an assumed distributions are plotted against the quantiles of the empirical distribution. If the sample is a realisation from the reference distribution the QQ-plot is approximately linear. In EVT, the GPD plot is typically confronted with the Exponential plot (an EVD for thin-tailed distributions).
The Hill plot (see Embrechts et al, 1997) is instead adopted in order to assess the validity of the shape parameter estimate and relies on the Hill estimator for fat tailed EVD.
Typically these two additional analyses are performed in absence of definite evidence coming from the mean excess function analysis.
In Figure 5 we show the estimated mean excess function, for increasing threshold absolute value (right to left in the figure) over all negative returns recorded in the 500 business days before the 30th of July 2001.

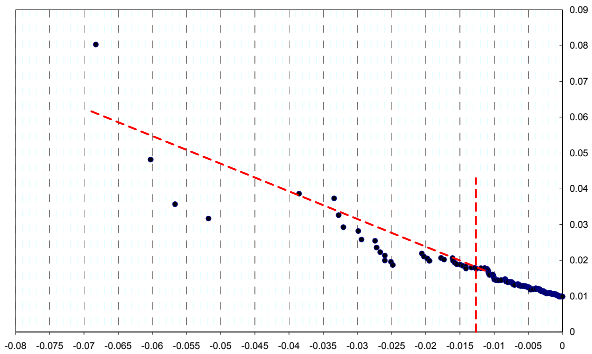

Figure 5. The mean excess function for Argentina JPM EMBI+ as at 30.7.01

The mean excess function increases linearly with respect to the threshold already for small negative values. As mentioned above this is a definite sign of tail thickness and justifies the adoption of the GPD as reference distribution on the *left* of the threshold.
A sufficiently high threshold needs however to be selected in order to reach a good fit of the distribution on the extreme tails through the Pareto limit distribution.
It is instructive to check the validity of the extreme value approximations before and after the sequence of events recorded in the market during the last three weeks of July 2001. This is relevant for the definition in RM practice of the VaR limits imposed to individual business units (B.U.) by the management. The stability of the approximating distributions facilitates that decision.
In table 1, we report the set of relevant parameters, estimated before (with closing price as at the 11th of July) and after the mid-July Argentinean crisis (prices as at the 30th of July). The data input is again represented by the 500 daily returns preceding the crisis. A sample of 300 daily returns is instead considered for the Gaussian model. The extended sample size in the EVT application is required in order to ensure a sufficient sample in the tail. For a more extended discussion on this point see Consigli (2002).
The comparative analysis is also relevant to assess the potential of the different parametric models in risk management.

JPM EMBI+ ARGENTINA		
	11.7.2001	30.7.2001
Gaussian		
mean	-0.0252	-0.1764
std deviation	0.170333469	0.249229774
GPD		
shape	0.2031	0.4087
scaling factor	0.0087	0.0093
Exponential		
lambda	0.0108	0.0171
threshold	-0.0099	-0.0106

Table 1. Parameters estimates *before* (11.7.01) and *after* (30.7.01) the July Argentinean turmoil

The –14.85% variation of the JPM Argentinean emerging market index on July the 12th induces a variation from –0.0099 to –0.0106 of the selected threshold, from which the sample set of excesses

for the EVD parameter estimation is derived. The increase of the shape parameter from 0.2031 to 0.4087 reflects the increasing heaviness of the left tail of daily returns.
Observe that the extreme event does induce a misleading, substantial, increase of the volatility estimate in the Normal distribution.
Here below in figure 6, we show the change of the GPD fit over the crisis period, associated with the above parameters. The lowest 1% percentile moves roughly from the –4% estimated on the 11th of July to the –5.5% at the end of the month. The robustness of the parameters estimates is reflected in the shape of the distribution.

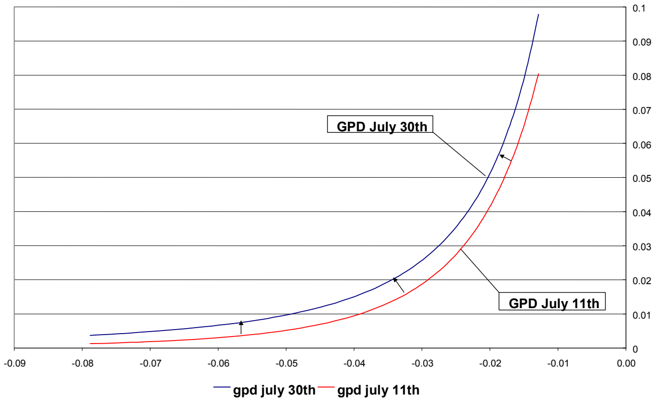

Figure 6. GPD fits over extreme return values before and after the Argentinean crisis

Endowed with the parameter estimates shown in the last column of table 1, we can approximate the empirical distribution of the tail negative returns on the Argentinean market and derive the required risk estimates.

Consider now the Poisson-Gaussian model.
The arrival of market shocks is captured within a mixed Poisson-Gaussian model. Unlike in the EVT-based method no threshold needs to be specified and risk estimation is simply based on the estimation of the distribution function of the p&l series.
We show in Table 2 the parameter estimates of both cases of mixed distribution with constant and lognormal jump sizes.
From the 300 daily returns history preceding July the 30th 2001, the following set of parameters on the corresponding models, have been estimated.

JPM EMBI+ ARGENTINA						
	11.7.2001			30.7.2001		
	Gaussian	Poisson+Gaussian fixed jump size	Poisson+Gaussian lognormal jump	Gaussian	Poisson+Gaussian fixed jump size	Poisson+Gaussian lognormal jump
mean	-0.11685	0.305881827	0.168278786	-0.11697	0.215291866	0.205017305
std deviation	0.19006	0.143141183	0.09798178	0.264751	0.200681981	0.099217835
Poisson intensity		13.92763927	53.69536869		5.242223432	51.45916476
jump average		-0.030440855	-0.005557263		-0.06408087	-0.00684218
jump variance			0.021543			0.031709078
ML value	900.6610422	930.7568208	961.9975804	801.2276	854.1770422	925.7306737
LRT	0	60.19155719	62.48151919	0	105.899	143.1072629

Table 2. Mixed Poisson-Gaussian model: parameters estimate before and after the July 2001 crisis.

The above parameters (expressed per year) define uniquely the probability distributions at the given date and allow the risk measures derivation.
Observe the significance in both cases of the Poisson component expressed by the LRT (likelihood ratio test). Also the quite remarkable drop of the ML optimal values between the two dates particularly for the Gaussian model. The introduction of the random jump parameter induces a significant improvement of the fit accuracy in both cases.
Relying on the above distribution specifications we can compare within a single plot different distribution fittings, focusing on July 30. We consider the following distributions: two in the family of the EVD, the GPD and the Exponential, fitted beyond the selected threshold of –1.1%, the two variations of the mixed Poisson-Gaussian model, with constant and lognormal jump size, and finally the benchmark Gaussian distribution against the empirical distribution. The input series is again represented by 300 daily variations of the JP Morgan market index.

It is shown in figure 6 that the normal distribution provides a very bad approximation both in the tails and the core of the support. The introduction of the Poisson component improves the fitting significantly for this case study as already clarified in table 2 above. Consistently with the behaviour of the mean excess function, the GPD provides a better fit than the exponential distribution beyond the –0.01 threshold.

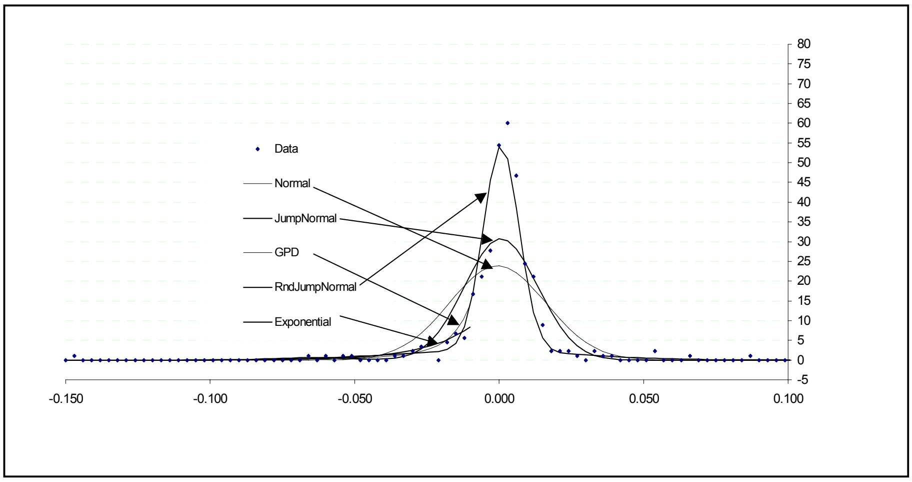

Figure 7. JPM Argentina EMBI+ estimated pdf's as at 30.7.2001

Next in figure 8, we zoom on the lower 10% percentile of the associated distribution functions.

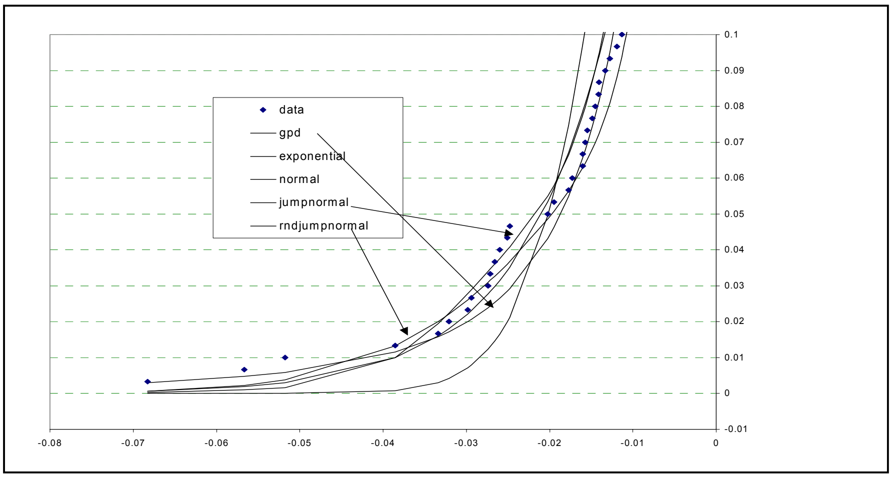

Figure 8. Lower 10% fit of the empirical distribution on the Arg bond market as at 30.7.2001.

The mixed Poisson-Gaussian distribution provides overall the best parametric proxy to the given empirical distribution.

All the estimated parametric distributions but the Normal, furthermore, assign even if very small a non null probability to the –15% return observed on July 12th.

Continuing the discussion on the Argentinean case study, we report in table 3, for the corresponding sample sets of Argentina EMBI+, the VaR and expected shortfall (ES) estimates for different percentiles collected before and after the crisis, again with the methods of historical simulation (HS), Gaussian (N), exponential (Exp), generalised Pareto distribution (GPD) and Poisson-Gaussian with constant (JN) and random (RJN) jumps.

11/07/2001	HS	N	JN	RJN	GPD	EXP
VaR(95%)	-1.6700%	-2.1100%	-2.1500%	-2.0000%	-1.8520%	-2.1191%
ES(95%)	n.a.	-2.5629%	-3.4017%	-3.2498%	-3.6446%	-3.2404%
VaR(99%)	-3.5900%	-2.8340%	-3.8000%	-4.1000%	-4.0792%	-3.7992%
ES(99%)	n.a.	-3.4645%	-5.5352%	-5.3848%	-6.5685%	-5.0908%
30/07/2001	HS	N	JN	RJN	GPD	EXP
VaR(95%)	-2.5000%	-2.7800%	-2.3200%	-2.6300%	-2.1964%	-2.6162%
ES(95%)	n.a.	-3.8790%	-5.0790%	-4.6440%	-4.2142%	-4.2941%
VaR(99%)	-5.8000%	-3.9000%	-6.6000%	-5.8500%	-5.4848%	-5.1349%
ES(99%)	n.a.	-4.8646%	-8.5118%	-7.3699%	-7.3630%	-7.1246%

Table 3. Return-at-Risk measures estimated in the Argentinean bond market across crisis period

Table 3 reports as columns the statistical method used to analyze the daily returns in the Argentinean market and as rows a set of relevant percentiles commonly used in risk analysis. Together with the VaR(95%-99%) we also report the associated mean excesses beyond the VaR. Excluding and including crisis day.

The table reports 1-day VaR(99%) risk measures expressed in percentage of the capital invested in this case in a representative market portfolio, typically adopted in RM practice to monitor internal market risk.

The Gaussian approximation in column N provides, as expected, an extremely poor approximation of the loss distribution.

The GPD method induces a less conservative approach than the exponential. It is however more accurate on the conditional moments when compared with the empirical distribution.

Benchmarking the historical simulation method with the GPD method gives very interesting results. Before the crisis the HS approach at the 99% level induces a more conservative strategy than the GPD, with a higher capital requirement coefficient. The opposite occurs after the crisis. The extent of the fall in the HS VaR(99%) is remarkable and gives a good example of the relative inefficiency of this measure for risk management purposes during and after a market shock. No indications can be derived within the HS approach on the shortfall distributions beyond the VaR estimates. The limited sample makes unbiased moment estimation impossible in this case.

The mixed Poisson-Gaussian approach (that in stable periods reduces to the simple variance covariance model) appears again very accurate and sensitive to the modified market conditions. Unlike the EVT-based method, that it is superior to the others far in the tail, not only the mixed approach is accurate with very limited extremes, but it does also provide a good approximation in the core of the distribution. This justifies the adoption of the model as reference model in unstable markets (see Consigli, 2002 also on this point).

The ES (CVaR) measure even with the above positive methodological remark, appears to be too expensive for capital allocation. It gives, however, a good description of the very extreme risk embedded in this market.

3.2.1 Method validation. Remarks

The previous analysis provides evidence of the ability of the Poisson-Gaussian model to capture events clustering leading to possibly multimodal density approximations. This model provides a viable alternative to EVT-based RM which was explicitly conceived to yield a more accurate tail approximation.

In general, as extensively discussed in Consigli (2002), acceptability in RM practice requires however the method to adapt to changing market conditions and extensive back-testing evidence based on the comparison between the percentile estimates induced by the given method and ex-post violation statistics.

These can be derived with respect to both VaR and ES estimates. In both cases the violations are defined by the differences between the risk measure estimated ex ante at the given confidence level and the observed market change.

In the VaR case, the percentage number of expected violations must be consistent with the adopted percentile (99%,95% confidence, etc.). The violation sizes, furthermore, are expected to be as small as possible, with a limited maximum violation. Both the number of registered violations and relevant statistics from the associated shortfall distributions are to be considered in the method validation procedure.

To give an example, we recall in the following figure, from Consigli (2002), the evidence collected for the Argentinean bond market in the two years between December 1999 and October 2001.

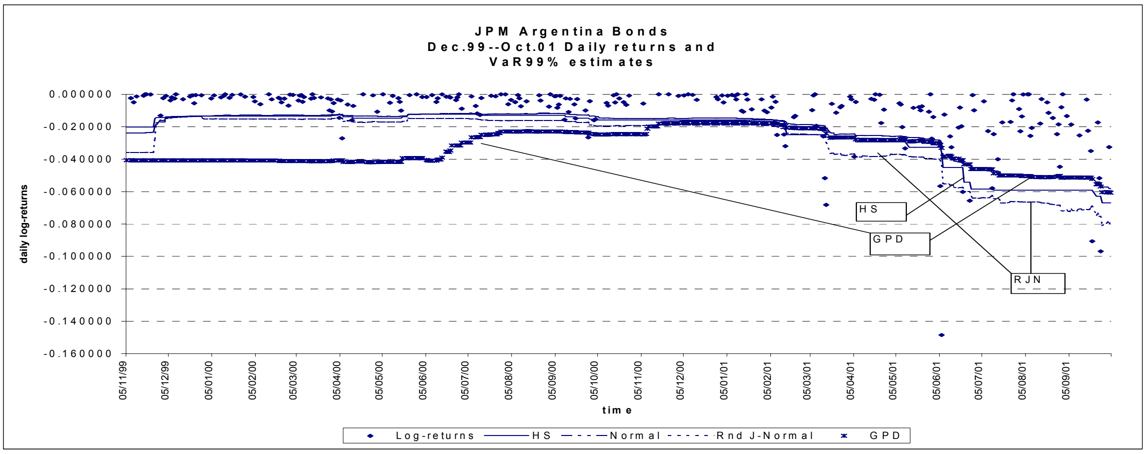

Figure 9. Back-testing results for the Argentinean bond market

Together with the statistics reported in that article the figure provides sufficient evidence of the validity of the mixed Poisson-Gaussian approach under different market regimes and its superiority with respect to HS and the EVT approach during the 2001 market instability. The extreme value method appears to be not sufficiently flexible and the positive remark on the stability of the GPD estimates for capital allocation turns into a negative feature when discussing the method accuracy.
The extension of the required sample set for the development of an EVT-based RM system has a negative impact in this sense.
The statistics associated with the ES estimation are less trivial to be derived but they can nevertheless be computed relying on the quantile ratios with respect to normal estimates (see Frey and McNeil, 1998).
An interesting evidence reported in last year study, had to do with the level of the VaR estimates in the different markets (see again Consigli, 2002). HS and Normal approximation tend to generate VaR estimates that on average are lower in the emerging market than in the two equity markets. The contrary holds true in the GPD case. This evidence confirms the limited impact that on the GPD estimates, unlike in the Gaussian case, is played by the market volatility. An evidence that tends to support an approach to VaR estimation based on the introduction of a jump component. In figure 8, the rapidity of the method to adjust to different market conditions is clearly shown. This is true also in presence of positive extreme market movements.
In conclusion we can recall the positive features of the mixed approach for risk estimation:

- The canonical Gaussian sub-case, when the Poisson process is estimated not to be significant
- The ability to provide good parametric tail approximations with limited sample sizes
- Its ability to capture, as in the above Argentinean case, the dynamics of the risk factors driving the bond prices in high yield (HY) markets. Several models have been proposed in the literature to account for possible markets shocks caused by default processes (see for instance Jarrow and Madan, 1995, Das, 1999, Duffie and Pan 1997)
- The possibility to introduce through the jump intensity and its amplitude ad-hoc shock models (see Consigli and Di Cesare 2001 and Madan and Unal, 1995)
- The possible extension, discussed in the next section, to a multivariate case and its adoptability in scenario-based portfolio optimisation.

3.3 Multidimensional Poisson-Gaussian model

The extension of the Poisson-Gaussian model to a multivariate system entails several difficulties. First: parameter estimation appears no longer attainable via simple maximum likelihood in the multidimensional case, second: the estimation of the Poisson correlations is difficult to achieve.
Few examples of such an attempt are present in the literature. Kim et al (1994) apply a Kalman filtering procedure to estimate the parameter matrix of a jump diffusion vector process.
Here we follow a simpler procedure in which a correlated system is built only on the continuous (diffusion) vector return process assuming independent Poisson arrivals and then introducing a heuristic method to account for possible shock propagation between different markets.

Under these assumptions it is possible to expand the classical Gaussian set-up, introducing a diagonal matrix of jump distributions and a vector of Poisson intensities.
The introduction of a specific tail correlation matrix, furthermore, allows, when desired (see § 3.1.1) the introduction of domino effects, a much relevant issue when dealing with financial crises and markets turmoil.
In presence of N risk factors and an equal number of Wiener and Poisson processes, respectively, we would have in the multivariate set-up, here expressed in uncorrelated form, the following system of stochastic equations:

$$\omega_t^i = \mu^i \Delta t + \sum_{j=1,..,N} \sigma^{ij} \Delta z_t^j + \sum_{k=1,...,N} \rho^{ik} \Delta N_t^k(\lambda^k) \qquad \omega_0^i = \omega \qquad i = 1,2,.,N \qquad (17).$$

System (17) describes the dynamic evolution[5] of daily returns: Δz_t^j and ΔN_t^k are respectively the independent increments of the j-th Brownian motion and the k-th Poisson process at time *t*. Not all intensities λ^k of the Poisson processes need to be positive (see below the case problem in § 4).
The drift vector μ and the volatility matrix σ are assumed constant. Again the jump magnitudes ρ^{ik} are random and distributed as lognormal random variables.
The vector $\omega_0 = \{\omega_0^i\}$ defines the initial known return vector.
System (17) can be represented in matrix form, introducing an (N,N) variance covariance matrix $\Sigma = \{\sigma^{ij}\}$, and an (N,N) diagonal matrix of random jump sizes $P = \{\rho^{ik}\}$:

$$\omega_t = \mu \Delta t + \Sigma \Delta z_t + \mathrm{P} \Delta N_t(\lambda) \qquad \omega(0) = \widetilde{\omega} \qquad (18).$$

In (18) ω_t is an N-dimensional random return process defined in (Ω, Ξ, P), the parameter space includes $\mu \in \Re^N, \Sigma \in \Re^{N\times N}, \mathrm{P} \in \Re^{N\times N}, \lambda \in \Re^N$: the drift vector, the variance covariance matrix, the jump magnitude matrix and the intensity vector, respectively, while Δz_t define the Brownian motion and ΔN_t the Poisson vector processes in the N-dimensional space.
The variance covariance matrix can also be expressed as the correlation matrix pre and post multiplied by the diagonal volatility matrix: $\Sigma = \sigma C \sigma$. In this form is now possible to take as elements of the drift μ, the diagonal volatility matrix σ, the jump matrix P and the vector of Poisson intensities λ, the estimated ML coefficients of the one dimensional case:

$$\omega_t = \mu^{ML} \Delta t + \sigma^{ML} C \sigma^{ML} \Delta z_t + \mathrm{P}^{ML} \Delta N_t(\lambda^{ML}) \qquad (19)$$

In the sequel we maintain the assumption of independent Poisson processes but introducing a possible propagation effect through the following heuristic, applied to the matrix of jump magnitudes and whose rationale is explained in the following section:

$$\mathrm{P} = \mathrm{T} \times \mathrm{P}^{ML} \qquad (20).$$

The elements of $\mathrm{T} := \{\tau^{li}\}_{l=i=N} \in [-1,1]^{N\times N}$ in (20) are the coefficients of tail correlation defined in § 3.3.1: the matrix contains a number of columns equal to the number of independent Poisson processes and a number of rows equal to the number of considered markets.
According to (20), assuming for instance a shock ρ^{ii} in market i (see eq. (18)), depending on its size and an estimated tail correlation τ^{li} with market l, we will have, if propagation is allowed, a jump of size $\rho^{li} = \tau^{li} \rho^{ii}$ in market l.
Model (19) represents the reference model for scenario generation in presence of market instabilities, for risk management and portfolio selection purposes.

[5] See Consigli and Di Cesare (2001) for the associated MC simulation algorithm.

3.3.1 An EVT-based heuristic for tail correlation analysis

We discuss in this section a possible extension of the analysis conducted above. The study of crises correlation is based on the introduction of a new process constructed as a ratio between the conditional positive mean excess and the negative mean excess.

The elements of the matrix $\mathrm{T} := \{\tau^{li}\}$ are the correlation coefficients between the following *extremal risk-return ratios,* defined at time $t = 1,2,...,$ and for market $i = 1,2,..,N$, by

$$I^i(t) = \frac{P^{i,up}(t,u)}{P^{i,down}(t,-u)}\left|\frac{ES^{i,up}(t)}{ES^{i,down}(t)}\right| \qquad (21).$$

In (21) $P^{i,up}(t,u)/P^{i,down}(t,-u)$ is the ratio between the probability of a positive return above u, the *positive excesses*, and the probability of a return below $-u$, the *negative excesses*. The threshold is here fixed on the negative returns at the 10% percentile: $P^{i,down}(t,-u) = 0.10.$

$ES^{i,up}(t)$ and $ES^{i,down}(t)$ are instead the average positive and negative expected shortfalls computed on the positive and negative tails: $ES^{i,up}(t) = \left[E_t\left(x\,|x > u\right)\right]$ and $ES^{i,down}(t) = \left[E_t\left(x\,|x < -u\right)\right]$.

$P^{i,up}(t)/P^{i,down}(t)$, the *probability ratio*, thus measures the relative frequency of positive against negative excesses. While $ES^{i,up}/ES^{i,down}$, the *mean excess ratio*, measures the average magnitude of those excesses.

The indicator is thus sensible to extreme returns both relative frequency and magnitude.

A positive extremal ratio, respectively a negative, denotes the presence of a statistically significant upside, respectively downside, in the extreme returns. The ratios tend to adapt to changing volatility, being based on the definition of the lowest 10% return percentile.

The ratio will increase when an increasing number of positive excesses is recorded and the resulting average excess either increases or remains constant, or does not decrease to such an extent as to offset the increased frequency.

Conversely the ratio will decrease.

The ratio stays constant if either no extreme movements are registered in the market (the general case, returns stay roughly within the 80% of the distribution), or the increased frequency of either positive or negative returns is offset by corresponding reductions of the associated mean excesses.

If, furthermore, for two different markets, we observe diverging extreme indicators, then at the time one is cumulating positive extreme upside the other is cumulating negative extreme downside: in the two markets extremes are thus negatively correlated. A crucial evidence, during trouble periods.

The ratios provide extreme return measures computed over a period of 500 days.

For our reference Argentinean market the ratio behaviour during year 2001 is shown in the following figure 10.

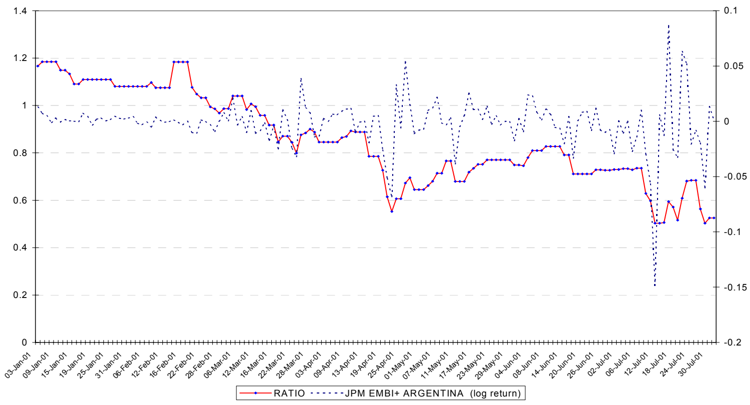

Figure 10. Year 2001 JPM EMBI+ Argentina extremal ratio (left scale) and daily log returns (right scale)

At the end of July, the Argentinean ratio was equal to 0.526. Due to a probability ratio of 0.600 multiplied by 0.876, the ratio between the positive and the negative mean excesses.

The indicator became lower than 1 on the last week of March 2001. At that point the market was on average already penalising investors looking for extremes.
A similar sequence of ratio estimates can be constructed for any other market and the correlation between the two over a 100 day moving window defines one entry of T.
The introduction of these correlations in a system in which the significance of the Poisson variable in individual markets has already been tested, is justified by the possibility of market movements beyond those already observed in the market, at which a propagation effect might be attached. Typically a high threshold (e.g. $\pm 3\sigma$) is introduced, above which a crisis propagation, is allowed.
This plays the role of a stress-testing exercise in classical RM practice.
The assumption underlying the introduction of $\mathrm{P} = \mathrm{T} \times \mathrm{P}^{ML}$ in (19) is that the correlation values, estimated according to (21) only on the tails and outside (before) periods of instability, will hold during instability periods. A similar evidence is discussed in Silvapulle and Granger (2001), Mikosh and Stărică (2000), Sancetta and Satchell (2003). A more detailed description of two case studies, related to the 1998 Russian crisis and the 2001 Argentinean crisis, can be found in Consigli et al (2001). The tail correlation matrix T and the correlation matrix of daily returns estimated during the two weeks instability periods are compared on a selected set of equity and bond markets.
We find the following relevant evidences:

- Negative correlations between the ratios before the crises stay all negative during the crises: This is what financial managers are looking for during periods of financial instability.
- Positive correlation during the crisis is anticipated by positive correlation between extremal ratios before the events
- The correlation ordering holds across the pre-crises and crises periods

Concluding: even on a limited sample, results are encouraging. Tails correlation, captured by the extremal ratios, tend to show a high degree of persistence during periods of financial instability and may provide good indications on forthcoming extreme market movements.
More statistical analysis must be performed in order to confirm the previous results and derive rules within a framework, where typically no rules exist and investors are wondering what is coming next.

4 Risk control based on portfolio optimisation

In the remaining part of this work we analyse the implications recalled in the introductory section 2, on risk control via portfolio optimisation methods, when two substantial changes are introduced in the classical risk-return framework:

- Returns are no longer considered joint normal, but, consistently with the results presented in § 3, we do consider a Poisson-Gaussian type of return generating process and
- Risk measures other than the variance of returns, specifically defined in the distribution negative tail, are considered.

Recall the general formulation of the portfolio optimisation problem given in (3). We now replace for notational convenience, the constraint formulation introducing the return function $R : X \times \Omega \rightarrow \Re$:

$$\min_{x \in X} \rho(x, \omega) \qquad s.t. \quad R(x, \omega) \geq r \qquad (22).$$

The decision space X can be characterised in order to generate different type of problems (see Consigli, 2002) Here we are rather interested in the solvability of the optimisation problem when arbitrary assumptions are allowed on $\omega \in (\Omega, \Xi, P)$ and different risk return pairs are introduced in the problem.
While, since the seminal work of Mandelbrot (1963), the research on more accurate modelling of financial returns has been quite central to the scientific debate, the introduction of new measures of risk in risk-reward portfolio models, has attracted increasing interest since the adoption of the VaR as a standard in RM practice. The reported evidence of the lack of convexity of the VaR with respect to portfolio rebalancing (see § 2) has further enhanced the effort in that direction.
In Consigli (2002) we presented an optimisation procedure allowing for more than one risk measure in the risk-return trade-off problem. The choice of the relevant measure becomes than an issue of risk preference and a possibly infinite number of coherent measures can be generated by taking any convex combination of coherent measures. The definition of the efficient choice space, however, needs in this case to be delimited by appropriate constraints on the risk parameters. We will report on this subject in the near future. Here below we instead extend the analysis considering explicitly the opportunities offered by other decision rules such as those mentioned in § 2.

Under Gaussian assumptions, like those already considered by Markowitz (1952) in his seminal contribution, the control on tail risk induced by the inclusion of a VaR objective in problem (3) can be analysed relying on the dependence of the VaR on the volatility and expected return of the associated portfolio, as already clarified in § 2.
An interesting set of results is available as far as Mean-VaR optimisation under Gaussian assumptions is concerned.
Alexander and Baptista (2000) show that under assumptions of returns joint normality, as the confidence interval at which the VaR is computed increases for fixed horizon, the minimum VaR portfolio converges to the minimum variance portfolio and the mean-VaR efficient frontier converges to the mean-variance efficient frontier.
Furthermore if a portfolio is mean-variance efficient, there must be a confidence interval at which that portfolio globally minimises the VaR.
Under normality the mean-VaR frontier is a proper subset of the mean-variance frontier (Wang, 2000). This comes from the linear relationship between VaR and expected return and variance in the Gaussian model. It is also true that the minimum VaR portfolio in the mean-VaR space dominates the minimum variance portfolio, that will not belong to the efficient frontier.

Other relevant approaches, moving away from the normality assumption, are those presented by Campbell et al (2001) under hypotheses of returns distributed as t-Student random variables. Lucas and Klaassen (1998) solve an optimisation problem under shortfall constraints with returns following also the the t-Student distribution.
Leibowitz and Kogelman (1991) provide an early reference on asset allocation with downside constraints.
More recently the interest in the field has been enhanced by the mentioned contribution by Rockafellar and Uryasev (2000) with the solution of the mean-CVaR problem. The CVaR, the expected value of the shortfall distribution beyond the VaR, is a coherent risk measure. It does implicitly constrain the VaR, being by definition higher than the VaR in the p&l space and can be solved by linear programming techniques. This model is taken as benchmark optimisation model for tail risk control in § 4.2.
The mean absolute deviation model (MAD, Konno and Yamazaki, 1991) discussed in § 4.1 as well as the mean-semideviation model (Speranza, 1993, Ogryczak and Ruszczynski, 1999) do also provide examples of risk-return selection models solvable applying linear programming (LP) techniques.
The crucial advantage coming from the representation of portfolio optimisation problems as LPs is that in this case any assumption on the underlying distribution can be considered. Scenario optimisation (see Zenios, 1993) can in this case be employed.
In § 4.2 we present a set of numerical results on alternative LP-solvable portfolio problems benchmarked against classical mean-variance and a fully diversified non optimal portfolio for the Argentinean crisis, with an estimated Poisson-Gaussian model of returns.
When the hypothesis of normality is removed, or more generally when no parametric approximation of the return distribution is considered, particularly during periods of financial instability, VaR cannot be recovered from the first two moments of the return distribution. Results joining the different risk minimisation problems can no longer be derived analytically in most cases.

4.1 Risk-return trade off optimisation: QP and LP solvability

We have mentioned above that a variety of problems formulation can be accommodated in (22) as we allow the risk measure to change. We present here very briefly a set of possible problem instances, that are then implemented and benchmarked in § 4.2.1 to yield comparative results on optimal static strategies during the Argentinean bond crisis.
The application of different portfolio paradigms and risk definitions in a period of financial distress, reflected in the adoption of a jump-diffusion model of financial returns, is aimed at analysing the potential of such schemes in controlling extreme downside risk. A similar analysis is conducted by Topaloglou et al (2002).
We expect in particular that portfolio problems taking explicitly into account the extreme downside will provide superior control at the time of market turmoil.
The mean-CVaR problem belongs to the general class of shortfall models in the case in which the shortfall is computed with respect to a random quantity. Its solution (see Rockafellar and Uryasev, 2000) is based on the introduction of an auxiliary variable accounting for the shortfall with respect to the associated VaR measure. The CVaR measure can be defined in the return space as the expected return beyond the value at risk at a given confidence interval. It is equivalent to the concept of the ES, as mentioned above.

The auxiliary variable included in the optimisation problem (see also Szegö, 2002) is defined by $y_\alpha^+(\omega) = (W(x,\omega) - VaR_\alpha(x) \wedge 0)$: this quantity is scenario dependent and the key idea of Rockafellar and Uryasev in (2002) is to introduce this shortfall variable both in the objective function -- $\min_x VaR_\alpha(x) + (1-\alpha)^{-1} E(y_\alpha^+)$ -- and the constraint region -- $y_\alpha^+(\omega) \geq \omega' x - VaR_\alpha(x)$ -- for all scenarios ω.

This modification of the extreme risk-return model allows the application of the LP approach. Again no conditions are imposed on the probability space in which ω is defined.

A similar class of problems can be generated by introducing as risk measure in (22) rather than the variance, the absolute deviation of the random final wealth from a given target $\widetilde{W}$. We have in this case the problem formulation

$$\begin{aligned} &\min_{x \in X} E\left[| W - \widetilde{W} | \right] \\ &s.t. \quad Ax = b \end{aligned} \qquad (23).$$

In (23) we have included all constraints in the matrix representation $Ax = b$: this can accommodate for the cash balance constraint, the expected return constraint and other structural constraints common in the formulation of portfolio problems. Problem (23) defines the so called *Mean absolute deviation* (MAD) model of Konno and Yamazaki (1991).

From (23) by constraining only the downside, we generate a second type of shortfall model, the *Mean Semi deviation* (MSD). With the same constraints the objective function becomes now:

$$\min_{x \in X} E\left[W - \widetilde{W} \wedge 0 \right] \qquad (24).$$

In (24) the decision maker seeks the minimisation of the negative deviations from the target wealth $\widetilde{W}$, without penalisation of the upside. In the case of a discrete number of scenarios for the random variable W, model (24) is a convex piecewise linear function of the scenario realisations, and is also LP computable.

The variance minimisation problem (4), observing that $\omega \in N(\mu, \Sigma) \Rightarrow W := \omega' x \in N(\mu' x, x' \Sigma x)$, generalises (23), when the variance around the mean return is penalised. This becomes with a quadratic utility formulation the Markowitz problem:

$$\min_{x \in X} E\left[W - \mu_W \right]^2 \qquad (25)$$

This formulation follows from:

$$x' \Sigma x = \sigma_W^2 = \sigma_W^2 + \mu_W^2 - 2\mu_W \mu_W + \mu_W^2 = E(W^2) - 2\mu_W EW + \mu_W^2 = E\left[W - \mu_W \right]^2.$$

From (25) it becomes apparent that we can solve a mean-variance type of problem by applying a QP algorithm once we have generated a sufficient sample of terminal wealth scenarios.

This can be done by assuming any probability distribution of financial returns. From finance theory we know that the solution of (24) parametric in the expected return will yield a mean-variance efficient frontier.

As mentioned above, in the following application we solve (25) by generating scenarios with the Poisson-Gaussian generator in (19).

By introducing a target portfolio value in (25) in place of the current portfolio expected value, we will induce a minimum variance about a target objective, that is also common in practice. Penalising only the downside we will generate a mean-semivariance problem. These type of problems are all solvable relying on QP methods.

Portfolio problems (23) to (25), unlike the mean-CVaR model, do not consider in their general formulation any one of the extreme tail measures previously discussed. The resulting optimal portfolios will however limit extreme losses by forcing portfolio rebalancing along those scenarios which deviate from the target scenarios. The generation of these scenarios, particularly during periods of financial turbulence, by the return process (19) should then yield effective risk control strategies.

We apply the above mentioned LP methods to solve in § 4.2.1 a risk control problem generated across the second phase of the Argentinean crisis (October 2001) for an extended set of equity and bond markets with different risk profile.

4.2 Optimal portfolios during periods of market instability

We can at this point summarise what has been argued up to now.

We started by describing a set of statistical methods leading to an accurate parametric approximation of the portfolio tail distribution, during and outside periods of instability. Methods that have proven sufficiently accurate during normal market conditions, have not proven robust during unstable periods. The mixed Poisson-Gaussian model, in this respect has shown a remarkable degree of flexibility under changing market conditions.
The introduction of this return generating model, however, by inducing the independence of the variance of portfolio returns from the risk measures defined in the tail of the distribution, calls for an explicit inclusion of the tail measures within the risk control framework. If the VaR is considered we have seen that serious convergence problems are induced by its non convex behaviour with respect to portfolio positions. This is a well known fact, for which however, in § 2, we have given a detailed graphical description. Alternative risk measures have been considered in § 4.1.
Furthermore the adopted random return model calls for a portfolio optimisation procedure based on a scenario representation of the uncertainty: we have seen that such an approach leads, for a rather significant set of risk measures to optimisation problems solvable by LP techniques.
Following the analysis in section 3, daily returns are assumed to follow a Poisson-Gaussian random process. By applying the same set of assumptions to risk estimation and a model portfolio optimisation, we relate, as suggested in the first part of this work, risk control to risk measurement within a coherent framework.
This framework is currently under development to yield a decision support tool linking Monte Carlo based parametric risk estimation to model portfolio optimisation for risk control in a computationally efficient platform. We will soon report on this development.
In this final part of the article we sum up by applying the described optimisation models to the case problem introduced in § 3.2. The presented results are not meant to be conclusive: we intend at this stage only to show the potentials offered by the adopted approach and the feasibility of an integrated risk measurement and control approach in presence of extreme risks.

4.2.1 A case study. Application 2: risk control

We present in this final section an application of the approach presented in the previous section to a risk control problem defined with an extended set of markets across the second Argentinean crisis, in the fall of 2001. During November 2001, after the instability already recorded in July, the market went into a definite condition of illiquidity and default was declared on the interest payments on all Eurobond issues. Political instability was also spreading at that point.
The series of recorded market shocks can be summarised. On Monday 29th of October the market measured by the JPM emerging market bond index suffered an initial already dramatic 9.06% fall. Then after the attempt to clear Argentinean positions world wide lead to the following sequence of market movements: -5,18% on Thursday 1st of November, -9,69% the day after, -8,38% on Friday 16th of November, -6,57% on the 27.11 and -10,17% on the 29th of November.
During this period unlike on the previous crisis wave, the crisis came at a moment in which markets were already suffering from the instability coming from the US equity market, after the September 11 terrorist attack. Markets were thus down trending and correlations were already positive across global equity markets. The optimisation problem is solved with an horizon set at the end of November 2001 and a here and now static decision to be taken on Friday the 26th of October.
The parameter vector was estimated on series of 300 daily returns for each market on that date in order to allow out-of-sample testing.
We summarise the implemented procedure:

- First perform ML estimation on the individual markets: this allows the definition of the markets with an embedded jump component
- Estimate the continuous time correlation matrix and the tail correlation matrix
- Simulate following system (19), 100,000 sample paths for every market: these are the scenarios entering the portfolio optimisation problem and on which out-of-sample VaR and ES back-testing can be performed
- Back-test for VaR violations at the 99% confidence level over the planning horizon
- Run scenario-based optimisation following different schemes and, finally
- Back-test risk control accuracy across the crisis period.

We consider a market span including the following set of global markets, expressed by corresponding market benchmarks of JP Morgan (JPM) and Bear Sterns (BS) for bond markets and stock exchange and Morgan Stanley (MSCI) fro equity markets:

- Bond markets: JPM US, JPM Germany, JPM UK, JPM JP, JPM ARG, JPM BRA, JPM Russia, BS HY
- Equity markets: S&P500, Nasdaq 100, Eurostoxx, Nikkey 225, MSCI Emerging

The portfolio problems are solved for a Euro-based investor through:

- Classical Mean-Variance optimisation
- Mean absolute deviation method with a 3% return benchmark at the 1 month horizon
- Mean semi deviation method with the same benchmark return
- Mean-CVaR optimisation: this is our benchmark extreme risk control approach.

The optimal portfolios generated through these models are benchmarked against the performance of a non optimal, fully diversified equally-weighted portfolio.

We present the following numerical evidences of the collected results: the initial matrix of ML estimates, the output of the scenario generator over the planning horizon for the Argentinean market only, the table of optimal portfolios with relative out of sample test.

Series	Model	drift p.y.	vol p.y.	shock freq	shock mean	shock vol	LRT	p-value
S&P500	StochJump	-0.164	0.169	125.778	-0.001	0.017	11.216	0.003668
Nikkei225	StochJump	-0.498	0.288	4.644	0.009	0.042	5.182	0.074945
Nasdaq100	StochJump	-1.883	0.515	15.672	0.074	0.017	15.464	0.000439
MsciEmerFree	StochJump	-0.184	0.204	10.047	-0.018	0.019	5.874	0.117904
JPMorganUSA	Gaussian	0.126	0.112	0	0	0	-	-
JpMorganUK	Gaussian	0.068	0.091	0	0	0	-	-
JpMorganRus	StochJump	0.456	0.163	64.809	-0.004	0.013	6.128	0.046701
JPMorganJapan	Gaussian	-0.058	0.14	0	0	0	-	-
JPMorganGer	StochJump	0.122	0.066	9.977	-0.002	0.036	15.082	0.001748
JpMorganBrz	Gaussian	-0.009	0.179	0	0	0	-	-
JpMorganArg	StochJump	-0.105	0.249	7.714	-0.009	0.062	63.298	1.8E-14
EuroStoox	StochJump	-0.228	0.198	20.665	-0.004	0.027	20.082	0.000163
BsHY	Gaussian	-0.029	0.125	0	0	0	-	-

Table 4. Maximum likelihood estimates as at 26.10.2001. Data input represented by 300 daily returns

The Argentinean market, where the instability originates from, is bolded and shows the presence of a significant Poisson component as was already established in first case study. As reported in the first row, for each market we indicate the adopted parametric model (see § 3), the estimated expected return and volatility per year, the intensity of the Poisson process and the mean and volatility of the jump magnitudes. As explained in § 3, the likelihood ratio test (LRT) and the associated p-statistics do give indication of the statistical significance of the jump variable. In 8 out of 13 markets there is evidence of a significant jump component. In five cases the LRT above 10 is a sign of definite statistical significance of the Poisson variable. We have pointed out that this was a period of generalised markets instability and this is accurately reflected in the estimated parameters.
The volatility of the shock variable for the Argentinean market captures part of the general market volatility over the period. Tail correlations among the different markets were generally positive but not high.
During year 2001 and part of 2002, bond markets were in general driven high by the observed reduction of US and European interest rates during a period of increasing fears of economic recession. This is reflected in the estimated parameters. The Russian bond market after the 1998 crisis, had another record year. At the time of the Argentinean crisis the Brazilian market did not suffer, but the crisis came few months afterward.
Together with the correlation matrix estimated on daily returns and the tail correlation matrix (with a threshold of +/- 3σ) described in § 3.3.1, the matrix defines the input of the scenario generator described in the vector system (19). Here below we show for economy of space the simulated trajectories of the Argentinean market only. T Figure 11 shows the simulated mean, mean plus and minus 2 the standard deviation and the worst and best simulated scenarios over 21 days representing one business month. The simulated paths are compared ex-post with the actual market behaviour. Inspite of a significant sequence of shocks, the worst case scenario captures the market fall during the first two weeks, while it does not during the second half of November.

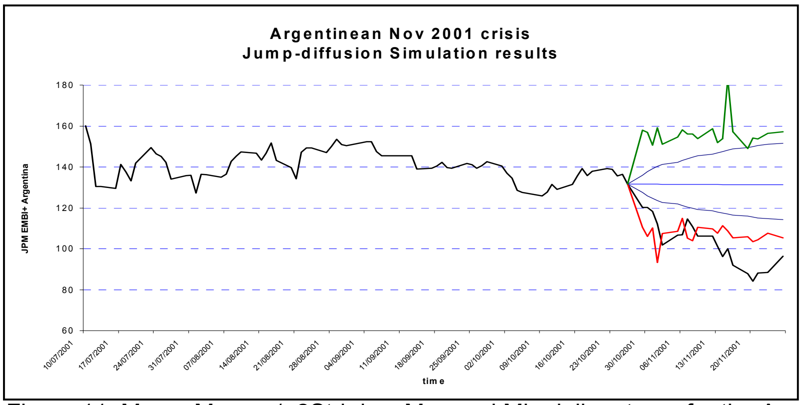

Figure 11. Mean, Mean +/- 2Std dev, Max and Min daily returns for the Argentinean market over the planning horizon. Out-of-sample testing of the scenario generator.

The slight recovery of the market at the end of November reduces at the month horizon the prediction error. On this market, as on the others not shown here, we do have consistent VaR(99%) and ES(99%) estimates at least up to the 10 day standard regulatory horizon.
The generated scenarios at the end of the month provide the input for the scenario based optimisation procedure.
Observed that even for the mean-variance canonical method the input drift vector and variance covariance matrix are those generated by the simulator, thus to a certain extent accounting for the presence of the Poisson return component.

OPT PFLIOS	Risk minimizing investors -- 1month horizon					end of
approach-->	bchmk	M/V diff	M/sDEV	M/CVaR	MADM	period
JpMorganBrz	0.076923	0	0	0	0.021091	0.1025
JpMorganArg	0.076923	0	0	0	0	-0.4167
JpMorganRus	0.076923	0.25	0	0	0	0.0737
JpMorganUK	0.076923	0	0.250000	0.250000	0.25	0.0093
Nasdaq100	0.076923	0	0	0	0.090036	0.0670
S&P500	0.076923	0	0	0	0	0.0519
EuroStoox	0.076923	0	0.000000	0.000000	0	0.0692
Nikkei225	0.076923	0	0.083065	0.083065	0	-0.0083
MsciEmerFree	0.076923	0	0	0	0	0.1043
JPMorganUSA	0.076923	0.25	0.250000	0.250000	0	-0.0004
JPMorganJapan	0.076923	0.25	0.138894	0.138894	0.25	-0.0013
JPMorganGer	0.076923	0.25	0.250000	0.250000	0.25	0.0079
BsHY	0.076923	0	0.028041	0.028041	0.138873	0.0519
E(return)	-0.00672	0.004013	0.008602	0.001827	0.001587	
Var(return)	0.0328	0.01386	0.019871	0.013545	0.015168	
VaR99%	0.083011	0.028225	0.037619	0.039679	0.043694	
Actual	0.8575%	0.5803%	2.0185%	0.479%	0.321%	

Table 5. Optimal portfolios across the Nov.2001 Argentinean crisis

All optimal problems have been run imposing a maximum 25% investment in any asset class. The rows report for every market the allocated percentage invested under the different methods and on the right the recorded end of the month market variation. For each optimal portfolio the expected return, the variance and the VaR99% estimate are reported together with the ex-post actual portfolio performance.
The mean-semideviation portfolio is the one with the best ex-post performance and its return is close to the 3% exogenous benchmark return. The fully diversified portfolio provides a return over the period higher than the M/V, the M/CVaR and the MAD portfolios. This is the portfolio however that over a more extended time span has shown by far the highest volatility and lack of excess return persistency.
We will soon report more extended results over rolling time windows, in and out periods of market instability, to derive more stable statistical evidence and indications on risk control effectiveness.
All optimal portfolios reported in the table do show in and out of sample a good degree of immunisation with respect to the incoming crisis.

5 Conclusions and future research

In this paper we have analysed the methodological implications and a set of relevant financial applications arising from the attempt to maintain a given set of assumptions on the random behaviour of financial markets when we try in practice to capture rare events probabilities within a risk measurement framework and generate, via portfolio optimisation, effective risk control strategies targeted at limiting extreme portfolio losses.

Several critical issues have been addressed in this setting and a sequence of relevant RM applications have been presented in conjunction with a detailed description of the required methodological steps and the current theoretical debate. In particular:

- The problems posed by the VaR measure in both RM and portfolio control have been highlighted and the need to overcome the use of this measure in practice has been pointed out from several perspectives. A progress that however requires a convergence between the views coming from the banking industry, the regulators and researchers in this area.
- The feasibility of a mixed Poisson-Gaussian approach to Monte Carlo based risk management during and outside instability periods. This has been discussed together with a relevant set of computational results on tail risk estimation, collected from other more commonly used RM methodologies. We have here slightly extended the analysis given in Consigli (2002), a work that can be considered companion to the present one.
- The opportunities offered by scenario-based optimisation techniques to allow for arbitrary assumptions on the probability measure adopted for risk estimation and the explicit introduction of extreme risk measures in portfolio optimisation problems. It is worth observing here that this framework naturally generalises to a dynamic setting, in which dynamic stochastic programming techniques can be employed (Consigli and Dempster, 1998). This is future research.

The discussion has been supported, as already done in Consigli (2002) by a set of computational results collected with respect to the 2001 Argentinean crisis. The evidence are not conclusive and further analysis is needed in order to conclude on the superiority of one form of control with respect to another: the mean-variance approach however, in view of its theoretical and practical limitations, appears unable to provide a viable model for tail risk control in presence of financial instability. The EVT approach instead, as discussed in § 3, even if appropriate for extreme risk measurement cannot be accommodated within a portfolio optimisation framework.

Forthcoming research will address:

- The inclusion of risk preferences in the decision framework and the opportunities offered by subjective-based risk definitions
- The potential offered by a dynamic formulation of the control problem
- The efficient design of a decision support system based on the techniques analysed in this report

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References

Acerbi,C. (2002), Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance* **26.7**, 1505—1518

Acerbi,C. and P.Simonetti (2002). Portfolio Optimization with Spectral Measures of Risk, Research paper downloadable from http://www.gloriamundi.org/

Acerbi, C. and D.Tasche (2001). On the coherence of Expected Shortfall. Working paper, downloadable from http://www.gloriamundi.org/

Alexander,G.J and A.M.Baptista (May, 2000). Economic implications of using a Mean-VaR model for portfolio selection: A comparison with Mean-Variance analysis, Research paper of the University of Minnesota (US) downloadable from http://www.gloriamundi.org/

Artzner, P., F.Delbaen, J-M. Eber and D.Heath (1997). Thinking coherently *RISK* **10**(11)

Artzner, P., F.Delbaen, J-M. Eber and D.Heath (1999). Coherent Measures of Risk, *Mathematical Finance* **9**, pp. 203—218

Balkema, A., L. De Haan (1974). Residual life time at great age, *Annals of Probability* **2**, pp. 792—804

Barone-Adesi, G., K.Giannopoulos e L.Vosper (March 2000). Filtering Historical Simulation: Backtest Analysis, downloadable from http://www.gloriamundi.org/

Basak,S. e A.Shapiro (2001). Value-at-Risk based Risk Management Optimal Policies and Asset Prices, *Review of Financial Studies,* Dec.2001, pp 371—405

Boudoukh,J., M.Richardson and R.F.Whitelaw (1998). The Best of Both Worlds: A Hybrid Approach to Calculating Value-at-Risk, *Risk* Magazine

Bouyé,H. V.Durrleman, A.Nikeghbali, G.Riboulet e T.Roncall (2000). Copulas for finance: A reading guide and some applications. WP, Group de Recherche Operationnelle, Credit Lyonnais Parigi, see http://grp.creditlyonnais.fr/content/rd/home_copulas.htm

Campbell, R., R.Huisman and K.Koedijk (2001). Optimal portfolio selection in a Value-at-Risk framework. *Journal of Banking & Finance 25*, pp. 1789—1804

Consigli, G (2003). Misure di rischio: un progresso possible? Quaderni del Dipartimento di T.E. e M.Q.S.P., Facoltà di Scienze Politiche, Università la Sapienza di Roma, Collana *Banche e Mercati Finanziari Internazionali, oggi* curata dal Prof.Szegö, Quaderno n. 27, pp. 1—25

Consigli,G (2002). Tail estimation and mean-VaR portfolio selection in markets subject to financial instability. *Journal of Banking and Finance* **26.7**, 1355—1382

Consigli G. and A. Di Cesare (2001). A simulation environment for discontinuous portfolio value processes. *Applied Stochastic Models in Business and Industry* **17**, pp.41—55

Consigli G., G. Frascella and G. Sartorelli (2001). Understanding financial markest with extreme value theory: from Value at Risk to crises correlation analysis. *UBM Research Series,* **3**, (Sept.2001) UniCredit Banca Mobiliare Research Department, Via T.Grossi 10, 20121 Milan.

Consigli, G. and MAH Dempster (1998). The CALM stochastic programming model for dynamic asset-liability management. In *World Wide Asset and Liability modelling.* J.M.Mulvey and W.T.Ziemba Eds. Cambridge University Press (1998), pp. 464—500

Das S.R. (1999). A direct discrete-time approach to Poisson-Gaussian bond option pricing in the Heath-Jarrow-Morton model. *Journal of Economic Dynamics and Control* **23**, pp 333—369

Danielsson, J, L. de Haan, L.Peng and C.G.De Vries (2001). Using a Bootstrap Method to choose the Sample Fraction in Tail Index Estimation. *Journal of Multivariate Analysis* **76**, pp.226—248

Danielsson, J and C.G.De Vries (2000).Value-at-Risk and Extreme returns. In Extremes and Integrated Risk Management, edited by Paul Embrechts, RISK Books in association with UBS Warburg, pp.85—106

Duffie, D. and J.Pan (1997). An overview of Value at Risk. *The Journal of Derivatives* (Spring:1997), pp. 7—49

Duffie, D. and J.Pan (2001). Analytical Value at Risk with Jumps and Credit risk. *Finance and Stochastics* **5**, pp.155—180

Eberlein,E. e U. Keller (1995). Hyperbolic Distributions in Finance in www.neyman.mathematik.uni-freiburg.de/homepages/eberlein/papers/eberlein.keller

Embrechts, P., C. Kluppelberg, T.Mikosch (1997). *Modelling extremal events for Insurance and Finance*, Springer-Verlag

Embrechts, P., A.McNeil e D.Straumann (1999) Correlation and dependence in risk management: properties and pitfalls, ETHZ research paper, see www.math.ethz.ch/finance

Fama, E.F. (1965). The behaviour of stock market prices. *Journal of Business of the University of Chicago,* **38**, pp. 34—105

Fisher, R.A. and L.H.C. Tippett (1928). Limiting Forms of the Frequency Distributions of the Largest or Smallest Member of a Sample. *Proceedings of the Cambridge Philosophical Society,* **24**, pp. 180—90

Frey, R. and A.J. McNeil (1998). Estimation of Tail-related Risk Measures for Heteroscedastic Financial Time Series: an Extreme Value Approach. ETHZ research paper, downloadable from www.math.ethz.ch/finance

Frey, R. and A.J. McNeil (2002). VaR and expected shortfall in portfolios of dependent credit risks: Conceptual and practical insights, *Journal of Banking and Finance* **26.7,** 1317—1334

Gaivoronski, A. and Georg Pflug (July 2000). Value at Risk in portfolio optimization: properties and computational approach. Working paper NTNU , Department of Industrial Economics and Technology Management, Trondheim, Norway

Gnedenko, B.V. (1943). Sur la Distribution limite du Terme maximum d'une Serie aleatoire. *Annuals of Mathematics,* **44**, pp. 423—53

Grootveld, H. and W.G.Hallerbach (2003). Upgrading the Value-at-Risk from diagnostic metric to decision variable: a wise thing to do? This volume

Hosking, J. and Wallis, J. (1987), Parameter and quantile estimation for the generalised Pareto distribution, *Technometrics* **29**, 339—349

Jarrow R.A. and D.Madan (1995). Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Mathematical Finance* **5.4**, pp. 311—336

Jorion, P. (1997). *Value-at-Risk: The new benchmark for controlling market risk*. McGraw-Hill (NY).

JP Morgan (1995a). Introduction to RiskMetrics, 3rd Edition, Market Risk Research, Morgan Guranty Trust (NY)

JP Morgan (1995b). Technical Document, 3rd Edition, Global Research, Morgan Guranty Trust (NY)

Kendall, M.G. (1953). The analysis of economic time series. Part I: Prices. *Journal of the Royal Statistical Society,* **96A**, pp. 11—25

Kijima, M, and Y.Muromachi (2000). Evaluation of credit risk of a portfolio with stochastic interest rate and default processes. *The Journal of Risk* **3.1**, pp.5--22

Kim, M-J., Y-H Oh and R.Brooks (1994). Are jumps in stock returns diversifiable? Evidence and implications for option pricing. *Journal of Financial and Quantitative Analysis*, Vol.**29**-No.4, pp. 609—631

Konno, H., H.Yamazaki (1991). Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market. *Management Science* **37**, pp.519—531

Krokhmal, P., J.Palmquist and S.Uryasev (2001). Portfolio optimisation with conditional Value-at-Risk objective and constraints. Research paper of the University of Florida, Risk Management and Financial Engineering (RMFE) Lab. Corresponding author S.Uryasev, RMFE Univ.of Florida, uryasev@ise.ufl.edu, http://www.ise.ufl.edu/uryasev

Leibowitz, M.L. and S.Kogelman (1991). Asset allocation under shortfall constraints. *Journal of Portfolio Management* **17**, pp. 18—23

Lucas, A. and P.Klaassen (1998). Extreme returns, downside risk and optimal asset allocation. *Journal of Portfolio Management* **25**, pp. 71—79

Luenberger,D.G. (1989). *Linear and Nonlinear Programming*, Addison-Wesley Pbl

Madan, D.B., and H.Unal (1995). Pricing the risks of default. *Working paper* **95.3**, College of Business, University of Maryland (US).

Mandelbrot, B. (1963). The Variation of certain speculative prices. *Journal of Business of the University of Chicago,* **36**, pp. 394—419

Markowitz, H.M. (1952). Portfolio selection, *Journal of Finance* **7(1)**, pp. 77—91

Matthys, G. and J.Beirlant (2000). Adaptive Threshold Selection in Tail Index Estimation. In *Extremes and Integrated Risk Management*, edited by Paul Embrechts, RISK Books in association with UBS Warburg, pp. 37—49

Mc Neil, A (2000). Extreme Value Theory for Risk Managers. In *Extremes and Integrated Risk Management,* edited by Paul Embrechts, RISK Books in association with UBS Warburg, pp. 3—19

Medova E.A. and M.N.Kyriacou (2002). Extremes in operational risk management. In *Risk Management: Value at Risk and Beyond.* MAH Dempster (ed). Cambridge University Press.

Merton, R.T. (1976). Option pricing when the underlying returns are discontinuous. *Journal of Financial Economics***,** **3**, pp. 125—144

Mikosch, T and C.S.Stărică (2000). Limit theory for the sample autocorrelations and extremes of a Garch(1,1) process. *The Annals of Statistics* **28,** pp. 1427—1451

Ogryczak, W and A.Ruszczynski (1999). From stochastic dominance to mean-risk models: semideviations as risk measures. *European Journal of Operational Research* **116**, pp.33—50
Pflug, G. (2000). Some remarks on the Value-at-Risk and the conditional Value-at-Risk. In *Probabilistic Constrained Optimization: Methodology and Applications,* S.Uryasev Ed., Kluwer Academic Pbl.
Pickands, J. (1975). Statistical Inference Using Extreme Order Statistics. *Annals of Statistics,* **3**, pp. 119—31
Rockafellar, R.T. (1970). *Convex Analysis*, Princeton Mathematics, Vol. **28**, Princeton University Press
Rockafellar, R.T. and S.Uryasev (2000). Optimization of Conditional Value-at-Risk. The Journal of Risk, Vol.**2**-No.3, pp. 21—41
Sancetta, A. and S.E.Satchell (2003). Changing correlation and Portfolio diversification failure in the presence of large market losses, *Research Paper,* Department of Applied Economics, University of Cambridge, Cambridge (UK)
Silvapulle, P and C.W.J.Granger (2001). Large returns, conditional correlation and portfolio diversification: a value-at-risk approach, *Quantitative Finance* vol.**1**, IOP Pbl, pp. 542—551
Speranza, M.G. (1993) Linear Programming models for portfolio optimization. *Finance* **14**, pp.107—123
Szegö, G. (2002). Measures of risk, *Journal of Banking and Finance* **26.7**, pp.1253—1272
Tasche, D. (2002). Expected shortfall and beyond, *Journal of Banking and Finance* **26.7**, pp.1519—1534
Topaloglou, N., H.Vladimirou and S.A.Zenios (2002). CVaR models with selective hedging for international asset allocation, *Journal of Banking and Finance* **26.7**, pp.1535—1561
Wang, J, (2000). Mean-variance-VaR Based Portfolio optimisation. *Working Paper* Department of Mathematics and Computer Science, Valdosta State University, Valdosta, GA, (US) Downloadable from www.gloriamundi.org
Zenios, S.A. (1993). *Financial Optimization*. Cambridge University Press, Cambridge (UK)